Graphs with Equal Irregularity Indices

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Abstract: The irregularity of a graph can be defined by different so-called graph topological indices. In this paper, we consider the irregularities of graphs with respect to the Collatz-Sinogowitz index [8], the variance of the vertex degrees [6], the irregularity of a graph [4], and the total irregularity of a graph [1]. It is known that these irregularity measures are not always compatible. Here, we investigate the problem of determining pairs or classes of graphs for which two or more of the above mentioned irregularity measures are equal. While in [17] this problem was tackled in the case of bidegreed graphs, here we go a step further by considering tridegreed graphs and graphs with arbitrarily large degree sets. In addition we present the smallest graphs for which all above irregularity indices are equal.

Keywords: irregularity measures of graph; topological graph indices

1 Introduction

Let $G$ be a simple undirected graph of order $n = |V(G)|$ and size $m = |E(G)|$. The degree of a vertex $v$ in $G$ is the number of edges incident with $v$ and it is denoted by $d_G(v)$. A graph $G$ is regular if all its vertices have the same degree, otherwise it is irregular. However, in many applications and problems it is of big importance to know how irregular a given graph is.

The quantitative topological characterization of irregularity of graphs has a growing importance for analyzing the structure of deterministic and random networks and systems occurring in chemistry, biology and social networks [7, 12]. In this paper, we consider four graph topological indices that quantify the irregularity of a graph. Before we introduce those indices, we present some necessarily notions and definitions.

A universal vertex is the vertex adjacent to all other vertices. We denote by $m_{r,s}$ the number of edges in $G$ with end-vertex degrees $r$ and $s$, and by $n_r$ the numbers of vertices in $G$ with degree $r$. Numbers $m_{r,s}$ and $n_r$ are referred as the edge-parameters and the vertex-parameters of $G$, respectively.
The mean degree of a graph $G$ is defined as $\overline{d}(G) = \frac{2m}{n}$. Graphs $G_1$ and $G_2$ are said to be edge-equivalent if for their corresponding edge-parameters sets $\{m_{r,s}(G_1) > 0\} = \{m_{r,s}(G_2) > 0\}$ holds. Analogously, they are called vertex-equivalent if for their vertex-parameters sets $\{n_r(G_1) > 0\} = \{n_r(G_2) > 0\}$ is fulfilled.

A sequence of non-negative integers $D = (d_1, d_2, \ldots, d_n)$ is said to be graphical if there is a graph with $n$ vertices such that vertex $i$ has degree $d_i$. If in addition $d_1 \geq d_2 \geq \cdots \geq d_n$ then $D$ is a degree sequence. The degree set, denoted by $D(G)$, of a simple graph $G$ is the set consisting of the distinct degrees of vertices in $G$.

The adjacency matrix $A(G)$ of a simple undirected graph $G$ is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position $(v_i, v_j)$ according to whether $v_i$ and $v_j$ are adjacent or not. The characteristic polynomial $\phi(G, t)$ of $G$ is defined as characteristic polynomial of $A(G)$:

$$\phi(G, \lambda) = \det(\lambda I_n - A(G)),$$

where $I_n$ is $n \times n$ identity matrix. The set of eigenvalues of the adjacent matrix $A(G)$ is called the graph spectrum of $G$. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the spectral radius of $G$. Graphs that have the same graph spectrum are called cospectral or isospectral graphs.

The four irregularity measures of interest in this study are presented next. The first one is based on the spectral radius of graph. If a graph $G$ is regular, then it holds that the mean degree $\overline{d}(G)$ is equal to its spectral radius $\rho(G)$.

Collatz and Sinogowitz [8] introduced the difference of these quantities as a measure of irregularity of $G$:

$$CS(G) = \rho(G) - \overline{d}(G).$$

The first investigated irregularity measure that depends solely on the vertex degrees of a graph $G$ is the variance of the vertex degrees, defined as

$$\text{Var}(G) = \frac{1}{n} \sum_{i=1}^{n} d_G^2(v_i) - \frac{1}{n^2} \left( \sum_{i=1}^{n} d_G(v_i) \right)^2.$$

Bell [6] has compared $CS(G)$ and $\text{Var}(G)$ and showed that they are not always compatible. Albertson [4] defines the irregularity of $G$ as

$$\text{irr}(G) = \sum_{u \neq v \in E} |d_G(u) - d_G(v)|.$$

In [1] a new irregularity measure, related to the irregularity measure by Albertson was introduced. This measure also captures the irregularity only by the difference of vertex degrees. For a graph $G$, it is defined as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u, v \in V(G)} |d_G(u) - d_G(v)|.$$
Very recently, \( \text{irr} \) and \( \text{irr}_t \) were compared in [9]. These irregularity measures as well as other attempts to measure the irregularity of a graph were studied in several works [2,3,5,13–15]. It is interesting that the above four irregularity measures are not always compatible for some pairs of graphs. The main purpose of this paper is to determine classes of graphs for which two or more of the above mentioned irregularity measures are equal.

The rest of the paper is organized as follows: In Section 2 we investigate tridegreed graphs that have equal two or more of the above presented regularity measures. In Section 3 we consider the same problem but for graphs with arbitrary large degree sets. The smallest graphs with equal irregularity measures are investigated in Section 4. Final remarks and open problems are presented in Section 5.

2 Tridegreed graphs

Most of the results presented in this section are generalized in Section 3. However, due to the uniqueness of the related proofs and used constructions, we present the results of tridegreed graphs separately.

2.1 An infinite sequence of tridegreed graphs with same \( \text{irr} \) and \( \text{irr}_t \) indices

Proposition 1. Let \( n \) be an arbitrary positive integer larger than 7. Then there exists a tridegreed graph with \( n \) vertices \( J(n) \) for which \( \text{irr}(J(n)) = \text{irr}_t(J(n)) \) holds.

Proof. The graph \( J(n) \) can be constructed as \( J(n) = C_{n-3} + P_3 \), where \( C_{n-3} \) is a cycle on \( n - 3 \) vertices and \( P_3 \) is a path on 3 vertices. It is easy to see that the graph obtained is tridegreed if \( n \) is larger than 7, and it contains one universal vertex, exactly. The vertex degree distribution of \( J(n) \) is \( n_5 = n - 3 \), \( n_{n-2} = 2 \) and \( n_{n-1} = 1 \). It can be shown that for \( J(n) \) the equality \( \text{irr}(G) = \text{irr}_t(G) \) holds. As an example graph \( J(9) \) is depicted in Figure 1.

It is easy to show that for graph \( J(9) \) the corresponding edge parameters are: \( m_{5,5} = 6, m_{5,7} = 12, m_{5,8} = 6, m_{7,8} = 2 \). Moreover, the equality \( \text{irr}(J(9)) = \text{irr}_t(J(9)) = 44 \) holds.

2.2 Pairs of tridegreed graphs with same \( \text{irr}, \text{irr}_t \) and \( \text{Var} \) indices

Theorem 1. Let \( G_a \) and \( G_b \) be connected edge-equivalent graphs. Then the equalities \( \text{irr}(G_a) = \text{irr}(G_b), \text{irr}_t(G_a) = \text{irr}_t(G_b) \) and \( \text{Var}(G_a) = \text{Var}(G_b) \) hold.
Proof. By definition, $\text{irr}(G)$ depends solely on the edge parameters of $G$. Since graphs $G_a$ and $G_b$ have same edge parameters, it follows that $\text{irr}(G_a) = \text{irr}(G_b)$. From the definitions of $\text{irr}_t(G)$ and $\text{Var}(G)$ indices, we have

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d(u) - d(v)| = \sum_r \sum_{s < r} n_r n_s (r - s),$$

$$\text{Var}(G) = \frac{1}{n} \sum_{u \in V(G)} d^2(u) - \left(\frac{2m}{n}\right)^2 = \frac{1}{n} \sum_r n_r \left(r - \frac{2m}{n}\right)^2.$$

So, $\text{irr}_t(G)$ and $\text{Var}(G)$ depend only on the vertex parameters of $G$. Since

$$r \cdot n_r(G) = \sum_{s \neq r} m_{r,s} + 2m_{r,r},$$

it follows that if graphs $G_a$ and $G_b$ are edge-equivalent, then $G_a$ and $G_b$ are necessarily vertex-equivalent as well, that is they have identical vertex-parameter set. Then, it also holds that $\text{irr}_t(G_a) = \text{irr}_t(G_b)$ and $\text{Var}(G_a) = \text{Var}(G_b)$. □

In Figure 2, two infinite sequences of pairs of tridegreed planar graphs that satisfied Theorem 1 are depicted. For a fixed integer $k \geq 1$, the graph $G_a(k)$ contains $k$ hexagons, while the graph $G_b(k)$ contains $k$ quadrangles. Graphs $G_a(k)$ and $G_b(k)$ have identical edge-parameters: $m_{3,1} = 4$, $m_{3,2} = 4k$, $m_{3,3} = k + 1$, and $n = 4k + 6$, $m = 5k + 5$. This implies that $G_a(k)$ and $G_b(k)$ have identical irregularity indices $\text{irr}$, $\text{irr}_t$ and $\text{Var}$. In what follows, we will verify that the converse of Theorem 1 is not true.

**Proposition 2.** There exist tridegreed connected graphs with different edge-parameter distributions but identical $\text{irr}_t$, $\text{Var}$ and CS irregularity indices.

Proof. An example is given in Figure 3. It is easy to see that polyhedral graphs (nanohedra graphs) depicted in Figure 3 are characterized by the following fundamental properties:

i) Polyhedral graphs $G_c$ and $G_d$ have $n = 8$ vertices and $m = 15$ edges.
ii) They have the same degree distribution: \( n_3 = 4, n_4 = 2 \) and \( n_5 = 2 \). This implies that \( \text{Var}(G_c) = \text{Var}(G_d) \), and their total irregularity indices are equal, \( \text{irr}_t(G_c) = \text{irr}_t(G_d) \).

iii) Their edge-parameter distributions are different, namely for graph \( G_c(m_{33} = 1, m_{34} = 4, m_{35} = 6, m_{45} = 4) \) and for graph \( G_d(m_{34} = 6, m_{35} = 6, m_{45} = 2, m_{55} = 1) \).

iv) Their Albertson indices are equal, \( \text{irr}(G_c) = \text{irr}(G_d) = 20 \). (This is an interesting fact, because the edge-parameter distributions of graphs \( G_c \) and \( G_d \) are different).

v) \( G_c \) and \( G_d \) are isospectral graphs (polyhedral twin graphs) [16]. This implies that their Collatz-Sinogowitz indices are equal, as well. \( \square \)
2.3 Pairs of tridegreed graphs with same irr, irr\_t, Var and CS indices

First, we state some necessary definitions and results needed for the derivation of the main results of this section. A bipartite graph $G$ is \textit{semiregular} if every edge of $G$ joins a vertex of degree $\delta$ to a vertex of degree $\Delta$. The 2-degree of a vertex $u$, denoted by $d_2(u)$, is the sum of degrees of the vertices adjacent to $u$ [20]. The average-degree of $u$ is $d_2(u)/d(u)$ and it is denoted by $p(u)$. A graph $G$ is called \textit{pseudo-regular} (or \textit{harmonic}) if every vertex of $G$ has equal average-degree. A bipartite graph is called \textit{pseudo-semiregular} if each vertex in the same part of a bipartition has the same average-degree [20]. It follows that semiregular graphs form a subset of pseudo-semiregular graphs.

**Theorem 2** ([20]). Let $G$ be a connected graph with degree sequence $(d_1, d_2, \ldots, d_n)$. Then

$$\rho(G) \geq \sqrt{\frac{d_2(v_1)^2 + d_2(v_2)^2 + \cdots + d_2(v_n)^2}{d_1^2 + d_2^2 + \cdots + d_n^2}},$$

with equality if and only if $G$ is a pseudo-regular graph or a pseudo-semiregular graph.

The following result is a consequence of Theorem 2.

**Corollary 1** ([20]). Let $G$ be a pseudo-regular graph with $d_2(v) = p \cdot d(v)$ for each $v \in V(G)$, then $\rho(G) = p$.

**Theorem 3.** There are infinitely many pairs of tridegreed pseudo-regular graphs $(G_1, G_2)$ for which $\text{irr}(G_1) = \text{irr}(G_2)$, $\text{irr}_t(G_1) = \text{irr}_t(G_2)$, $\text{Var}(G_1) = \text{Var}(G_2)$, and $\text{CS}(G_1) = \text{CS}(G_2)$.

**Proof.** We prove the theorem by a construction. Let $G(2, x, y)$ be a graph with vertex set $V(G(2, x, y)) = U \cup W \cup \{z\}$ with connectivity determined as follows: vertex set $U = \{u_1, u_2, \ldots, u_x\}$ induces connected 2-regular subgraph (cycle $c_1c_2 \cdots c_x$); $W$ is comprised of $y \cdot x$ pendant vertices such that each vertex from $U$ is adjacent to $y$ vertices from $W$; and the ‘central vertex’ $z$ is adjacent to each vertex from $U$. Two instances of such graphs, $G_1 = G(2, 7, 1)$ and $G_2 = G(2, 13, 2)$, are depicted in Figure 4. The parameter $2$ in the graph’s representation indicates that the vertex set $U$ induces connected 2-regular subgraph. The average degree of a vertex from $U$ is $(2(3+y) + x + y)/(3+y)$. The average degree of a vertex from $W$ is $3+y$, which is also the average degree of $z$. Thus, $G(2, x, y)$ is pseudo-regular graph if $(2(3+y) + x + y)/(3+y) = (3 + y)$, or if

$$x = y^2 + 3y + 3.$$  \hfill (1)
Next, consider a pair of edges \((u_i u_{i+1}, u_j u_{j+1})\) such that \((i \mod x) + 2 < j\). We delete edges \(u_i u_{i+1}\) and \(u_j u_{j+1}\) and add edges \(u_i u_{j+1}\) and \(u_{i+1} u_j\) to \(G_1\), obtaining a graph \(G'(2, x, y)\), which is edge equivalent (and therefore vertex equivalent) to \(G(2, x, y)\). Also, the average degrees of the vertices of \(G'(2, x, y)\) are equal to the average degrees of the vertices of \(G(2, x, y)\). By Corollary 1, \(\rho(G(2, x, y)) = \rho(G'(2, x, y)) = 3 + y\), for infinitely many integer solutions \((x, y)\) of (1). This together with the fact that \(G(2, x, y)\) and \(G'(2, x, y)\) are
edge equivalent, gives that $G(2, x, y)$ and $G'(2, x, y)$ have equal irr, irr, Var and CS indices.

Two pairs of pseudo-regular graphs $G(2, 7, 1)$ and $G'(2, 7, 1)$, and pair $G(2, 13, 2)$ and $G'(2, 13, 2)$, for which Theorem 3 holds, are depicted in Figure 4. These graphs correspond to the first two smallest pairs of integers that solve the equation (1).

Observe that the class of pair of graphs that satisfies Theorem 3 can be extended by considering graphs $G(k, x, y)$, $k \geq 2$. These graphs are generalization of $G(2, x, y)$ graphs, in such a way that the vertex set $U$ induces a $k$-regular subgraph.

An alternative construction. Next, we will present a new construction, that asserts the claim of Theorem 3. This construction is based on so-called Seidel switching [19], which for a vertex $v$ flips all the adjacency relationships with other vertices, i.e., all of the edges adjacent to $v$ are removed and the edges that were not adjacent to $v$ are added. In general, for a subset $S$ of $V(G)$, the graph $H$ is obtained from the graph $G$ by switching about $S$ if $V(H) = V(G)$ and $E(H) = \{uv \in E(G)|u, v \in S \text{ or } u, v \notin S\} \cup \{uv \notin E(G)|u \in S \text{ and } v \notin S\}$.

Construction by local switching.[ 11] Let $G$ be a graph and let $\pi = (C_1, C_2, \ldots, C_k, D)$ be a partition of $V(G)$. Suppose that, whenever $1 \leq i, j \leq k$ and $v \in D$, we have

(a) any two vertices of $C_i$ have same number of neighbors in $C_j$, and
(b) $v$ has either 0, $n_i/2$ or $n_i$ neighbors in $C_i$, where $n_i = |C_i|$. 

The graph $G(\pi)$ formed by local switching in $G$ with respect to $\pi$ is obtained from $G$ as follows. For each $v \in D$ and $1 \leq i \leq k$ such that $v$ has $n_i/2$ neighbors in $C_i$, delete these $n_i/2$ and join $v$ instead to the other $n_i/2$ vertices in $C_i$.

The property of the above construction that will be used here is the following one.

Theorem 4 ( [11]). Let $G$ be a graph and let $\pi$ be a partition of $V(G)$ which satisfies properties (a) and (b) above. Then $G^{(\pi)}$ and $G$ are cospectral, with cospectral complements.

The following construction is a special case of the construction by local switching, and will be used to construct infinite series of pairs of graph with the property stated in Theorem 3.

An example of the construction by local switching. A graph $G$ is comprised of $k$-regular graph $H$ on even number of vertices and one additional vertex $v$
adjacent to exactly half of the vertices of $H$. For $\pi(V(H), \{v\})$, we have that $G(\pi)$ is obtained by joining $v$ instead to the other vertices of $H$.

In the above example, as it was mentioned in [11], if $H$ has $2m$ vertices and a trivial automorphism group, than all $\binom{2m}{m}$ possible realisations of $H$ are non-isomorphic. By Theorem 4 the graphs $G$ and $G(\pi)$ are cospectral. $G$ and $G(\pi)$ have also same degree set $D(G) = D(G(\pi)) = \{k, k + 1, m\}$. The number of edges with endvertices with degrees $m$ and $k$ in $G$ is the same as in $G(\pi)$. The same holds for edges with endvertices with degrees $m$ and $k + 1$, and $m$ and $m + 1$. Thus, $G$ and $G(\pi)$ are edge equivalent, and $\text{irr}(G) = \text{irr}(G(\pi))$, $\text{irr}_t(G) = \text{irr}_t(G(\pi))$, $\text{Var}(G) = \text{Var}(G(\pi))$ and $\text{CS}(G) = \text{CS}(G(\pi))$. Note that if $H$ has less than 8 vertices, then $G$ and $G(\pi)$ are isomorphic. In Figure 5 an example of Seidel switching for $H = C_8$ (cycle with 8 vertices) is depicted.

![Fig. 5. Seidel switching when $H$ is a cycle with 8 vertices](image)

### 3 Graphs with arbitrary large degree set and same irregularity indices

#### 3.1 An infinite sequence of graphs with same irr and irr<sub>t</sub> indices

A graph $G$ is a complete $k$-partite graph if there is a partition $V_1 \cup \cdots \cup V_k = V(G)$ of the vertex set, such that $uv \in E(G)$ if and only if $u$ and $v$ are in different parts of the partition.

**Proposition 3.** There is an infinite sequences of graphs $\mathcal{G}$, such that for a graph $G \in \mathcal{G}$ $\text{irr}(G) = \text{irr}_t(G)$ holds.

**Proof.** If every two vertices of $G$ with different degrees are adjacent, then $\text{irr}(G) = \text{irr}_t(G)$. Graphs that satisfy this condition are the complete $k$-partite graphs. \qed

#### 3.2 Pairs of graphs with arbitrary large degree set and same irr, irr<sub>t</sub>, and Var indices

**Proposition 4.** There are infinitely many graphs $G_1$ and $G_2$ with same arbitrary cardinality of their degree sets satisfying $\text{irr}(G_1) = \text{irr}(G_2)$, $\text{irr}_t(G_1) = \text{irr}_t(G_2)$, and $\text{Var}(G_1) = \text{Var}(G_2)$. 
Proof. Consider the graphs $G_{csl}^1(14, 2, 4)$ and $G_{csl}^2(14, 2, 4)$ depicted in Figure 6. The graphs are bidegreed edge-equivalent, belong to the so-called complete split-like graphs, and were introduced and studied in [17]. Choose vertices $u \in V(G_{csl}^1(14, 2, 4))$ and $u \in V(G_{csl}^2(14, 2, 4))$ such that $d(u) = d(v)$. Attach to $u$ an arbitrary graph $H$ obtaining a graph $G_1$. Attach to $v$ a copy of $H$ obtaining a graph $G_2$. The graphs $G_1$ and $G_2$ are also edge-equivalent and therefore, $\text{irr}(G_1) = \text{irr}(G_2)$, $\text{irrt}(G_1) = \text{irrt}(G_2)$, $\text{Var}(G_1) = \text{Var}(G_2)$. □

Observe that in the construction, presented in the above proof, one instead of $G_{csl}^1(14, 2, 4)$ and $G_{csl}^2(14, 2, 4)$ can use any edge-equivalent graphs, for example graphs $G_a(k)$ and $G_b(k)$ in Figure 2.

3.3 Pairs of graphs with arbitrary large degree set and same irr, irrt, Var and CS indices

The 0-sum of two graphs $G$ and $H$ is got by identifying a vertex in $G$ with a vertex in $H$. To obtain the result of this section, we will use the following theorem and a corollary of it.

Theorem 5 ([10]). Let $F$ be a 0-sum obtained by merging $v$ in $G$ with $v$ in $H$, then the characteristic polynomial of $F$ is

$$\phi(F, \lambda) = \phi(G, \lambda)\phi(H \setminus v, \lambda) + \phi(G \setminus v, \lambda)\phi(H, \lambda) - \lambda\phi(G \setminus v, \lambda)\phi(H \setminus v, \lambda).$$

Corollary 2 ([10]). If we hold $G$ and its vertex $v$ fixed, then the characteristic polynomial of the 0-sum of $G$ and $H$ is determined by the characteristic polynomials of $H$ and $H \setminus v$. 

(a) 

(b) 

Fig. 6. Bidegreed edge-equivalent complete split-like graphs, (a) $G_{csl}^1(14, 2, 4)$ and (b) $G_{csl}^2(14, 2, 4)$
Fig. 7. Two cospectral and edge-equivalent graphs $G_1$ and $G_2$ obtained as 0-sums of $H$ and arbitrary graph $G$

Theorem 6. There are infinitely many graphs $G_1$ and $G_2$ with same arbitrary cardinality of their degree sets satisfying $\text{irr}(G_1) = \text{irr}(G_2)$, $\text{irr}_t(G_1) = \text{irr}_t(G_2)$, $\text{Var}(G_1) = \text{Var}(G_2)$, and $\text{CS}(G_1) = \text{CS}(G_2)$.

Proof. Let $G$ be an arbitrary graph. Consider the graph $H$ in Figure 7. Let $G_1$ be a 0-sum of $H$ and $G$, obtained by merging $v$ in $G$ with $v$ in $H$, and $G_2$ be a 0-sum obtained by merging $u$ in $G$ with $u$ in $H$. Note that $H \setminus v$ and $H \setminus u$ are isomorphic, so $\phi(H \setminus v, \lambda) = \phi(H \setminus u, \lambda)$. Together with Corollary 2, we have that $\phi(G_1, \lambda) = \phi(G_2, \lambda)$, or that $G_1$ and $G_2$ are cospectral. Also, it is easy to see that $G_1$ and $G_2$ are edge-equivalent. Thus, $G_1$ and $G_2$ have same irr, $\text{irr}_t$, $\text{Var}$ and $\text{CS}$ indices. □

A generalization of the example from Figure 7 is given in Figure 8. The graph $H$ is comprised of three isomorphic subgraphs $Q_l$, $Q_m$, $Q_r$, each of order at least 3, and two vertices $u$ and $v$. Between the vertex $v$ and the subgraph $Q_r$, there are same number of edges as between the vertex $u$ and the subgraph $Q_m$. Also, between the vertex $u$ and the subgraph $Q_l$, there are same number of edges as between the vertex $v$ and the subgraph $Q_m$. The number of the edges between $v$ and subgraph $Q_m$ differs than the number of the edges between $v$ and subgraph $Q_r$. We require these conditions to avoid an isomorphism of graphs $G_1$ and $G_2$, obtained as 0-sums of $H$ and arbitrary graph $G$. The graphs $G_1$ and $G_2$ are constructed in the same manner as above: $G_1$ is a 0-sum obtained by merging $v$ in $G$ with $v$ in $H$, and $G_2$ be a 0-sum obtained by merging $u$ in $G$ with $u$ in $H$. From the construction it follows that $G_1$ and $G_2$ are edge-equivalent. In this case also $H \setminus v$ and $H \setminus u$ are isomorphic, so $\phi(H \setminus v, \lambda) = \phi(H \setminus u, \lambda)$. Together with Corollary 2, we have that $\phi(G_1, \lambda) = \phi(G_2, \lambda)$, or that $G_1$ and $G_2$ are cospectral. Thus, it
Fig. 8. A generalization of the example from Figure 7

holds that \( \text{irr}(G_1) = \text{irr}(G_2) \), \( \text{irr}_t(G_1) = \text{irr}_t(G_2) \), \( \text{Var}(G_1) = \text{Var}(G_2) \), and \( \text{CS}(G_1) = \text{CS}(G_2) \).

4 Small graphs with identical irregularities

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. We said that \( G_1 \) is smaller than \( G_2 \) if and only if \( |V_1| + |E_1| < |V_2| + |E_2| \). Consequently, for two pairs of graphs \( P_1 = (G_1, G_2) \) and \( P_2 = (G_3, G_4) \), we said that \( P_1 \) is smaller than \( P_2 \) if and only if \( |V_1| + |E_1| + |V_2| + |E_2| < |V_3| + |E_3| + |V_4| + |E_4| \). The results in this section are obtained by computer search using mathematical software package Sage [18].

Proposition 5. There are no two graphs, both of same order \( n \leq 5 \), that have identical irregularity indices \( \text{CS}, \text{Var}, \text{irr} \) and \( \text{irr}_t \).

Next the smallest example of pair of graphs will be given with equal \( \text{CS}, \text{Var}, \text{irr} \) and \( \text{irr}_t \) indices.

4.1 Graphs of order 6

The smallest pair of graphs with identical irregularity indices \( \text{CS}, \text{Var}, \text{irr} \) and \( \text{irr}_t \) is depicted in Figure 9. Both graphs are of order 6, but one is of size 6 and the other of size 9. Their \( \text{CS}, \text{Var}, \text{irr} \) and \( \text{irr}_t \) indices are 0.236068, 0.800000, 8, and 16, respectively. They have different spectral radii, namely
Fig. 9. The smallest pair of (tridegreed) connected graphs with identical irregularity indices $CS$, $Var$, $irr$ and $irr_t$

the smaller one has spectral radius 2.236068 and bigger one 3.236068. The rest of the graphs of order 6, with identical irregularity indices $CS$, $Var$, $irr$ and $irr_t$ are given in Figure 10. The parameters of the graphs of order 6 with

Fig. 10. Besides the pair in Figure 9, there are three other pairs of connected graphs of order 6 $(a), (b), (c)$, and only one triple of graphs of order 6 $(d)$ with identical irregularity indices $CS$, $Var$, $irr$ and $irr_t$

identical irregularity indices $CS$, $Var$, $irr$ and $irr_t$ are summarized in Table 1. The graphs are enumerated with respect to their sizes, a smaller graph has smaller associated number ($G_{no}$). For a given graph, beside the values of the
indices CS, Var, irr and irr_t, its spectral radius \( \rho \), degree sequence and graph6 code are given. There are 112 non-isomorphic connected graphs of order 6.

Table 1: All four pairs and the only triple of graphs of order 6 with identical irregularity indices CS, Var, irr and irr_t.

<table>
<thead>
<tr>
<th>Tuple_no</th>
<th>G_no</th>
<th>graph6</th>
<th>degree sequence</th>
<th>irr</th>
<th>irr_t</th>
<th>CS</th>
<th>Var</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>E?o</td>
<td>[3, 3, 2, 2, 1, 1]</td>
<td>8</td>
<td>16</td>
<td>0.236068 0.800000 2.236068</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>77</td>
<td>E[w]</td>
<td>[4, 4, 3, 3, 2, 2]</td>
<td>8</td>
<td>16</td>
<td>0.236068 0.800000 3.236068</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>36</td>
<td>EKNG</td>
<td>[3, 3, 2, 2, 2, 2]</td>
<td>4</td>
<td>8</td>
<td>0.080880 0.266667 2.414214</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>37</td>
<td>E'NG</td>
<td>[3, 3, 2, 2, 2, 2]</td>
<td>4</td>
<td>8</td>
<td>0.080880 0.266667 2.414214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>E7~o</td>
<td>[4, 4, 2, 2, 2, 2]</td>
<td>16</td>
<td>16</td>
<td>0.161760 1.066667 2.828427</td>
<td></td>
<td></td>
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<td></td>
<td>100</td>
<td>EK~w</td>
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<td>16</td>
<td>16</td>
<td>0.161760 1.066667 3.828427</td>
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<td>90</td>
<td>EK~o</td>
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<td>8</td>
<td>0.038948 0.266667 3.732051</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>E[w]</td>
<td>[5, 5, 4, 4, 4, 4]</td>
<td>8</td>
<td>8</td>
<td>0.038948 0.266667 4.732051</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>54</td>
<td>Elmo</td>
<td>[3, 3, 3, 2, 2, 2]</td>
<td>4</td>
<td>8</td>
<td>0.065384 0.266667 2.732051</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>55</td>
<td>Ejeg</td>
<td>[3, 3, 3, 2, 2, 2]</td>
<td>4</td>
<td>8</td>
<td>0.065384 0.266667 2.732051</td>
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<tr>
<td></td>
<td>103</td>
<td>Ejmw</td>
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<td>4</td>
<td>8</td>
<td>0.065384 0.266667 3.732051</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.2 Graphs of order 7

There are 853 non-isomorphic connected graphs of order 7. The pairs of graphs of order 7 with identical irregularity indices CS, Var, irr and irr_t are given in Table 2. The smallest pair of connected graphs of order 7 with identical irregularity indices is depicted in Figure 11.

Table 2: All pairs of graphs of order 7 with identical irregularity indices CS, Var, irr and irr_t.

<table>
<thead>
<tr>
<th>Pair_no</th>
<th>G_no</th>
<th>graph6</th>
<th>degree sequence</th>
<th>irr</th>
<th>irr_t</th>
<th>CS</th>
<th>Var</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>104</td>
<td>FK?O</td>
<td>[3, 3, 2, 2, 2, 2]</td>
<td>6</td>
<td>10</td>
<td>0.057209 0.238095 2.342923</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>F'O</td>
<td>[3, 3, 2, 2, 2, 2]</td>
<td>6</td>
<td>10</td>
<td>0.057209 0.238095 2.342923</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>177</td>
<td>FAerO</td>
<td>[3, 3, 3, 3, 2, 2]</td>
<td>6</td>
<td>12</td>
<td>0.097508 0.285714 2.641186</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>178</td>
<td>FAeO</td>
<td>[3, 3, 3, 3, 2, 2]</td>
<td>6</td>
<td>12</td>
<td>0.097508 0.285714 2.641186</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>213</td>
<td>FK?W</td>
<td>[4, 4, 2, 2, 2, 2]</td>
<td>12</td>
<td>20</td>
<td>0.242178 0.952381 2.813607</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>214</td>
<td>F'WO</td>
<td>[4, 4, 2, 2, 2, 2]</td>
<td>12</td>
<td>20</td>
<td>0.242178 0.952381 2.813607</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>244</td>
<td>F?vw</td>
<td>[4, 4, 2, 2, 2, 2]</td>
<td>16</td>
<td>24</td>
<td>0.217713 1.142857 3.074856</td>
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<td></td>
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<tr>
<td></td>
<td>269</td>
<td>FA~o</td>
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<td>16</td>
<td>24</td>
<td>0.217713 1.142857 3.074856</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>274</td>
<td>F@VW</td>
<td>[4, 4, 3, 3, 2, 2]</td>
<td>12</td>
<td>22</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>275</td>
<td>F@UW</td>
<td>[4, 4, 3, 3, 2, 2]</td>
<td>12</td>
<td>22</td>
<td>0.173179 0.809524 3.030322</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 11. The smallest pair of connected graphs of order 7 with identical irregularity indices CS, Var, irr and irr \( t \) (the pair 1 in Table 2)
We would like to note that there is no triple of graphs of order 7 with identical irregularity indices CS, Var, irr and irr_t.

5 Conclusion and open problems

We have studied four established measures of irregularity of a graph. In particular, we have considered the problem of determining pairs or classes of graphs for which two or more of the purposed measures are equal. Some related results in the case of bidegreed graphs were presented in [17]. Here we have extended that work for tridegreed graphs and graphs with arbitrarily large degree set.

In the investigations here, it was assumed that considered graphs are of the same order, or they even have same degree sets. With respect to that, there are several interesting extension of the work done here.

It would be of interest to determine graphs of same order which have different degree sets, but their corresponding irr_t and irr indices are identical. A graph pair of such type with 5 vertices is illustrated in Figure 12. Also, it would be of interested to find classes of graphs of different order with equal irregularity measures. Most of the result presented have involved only pairs of graphs. Extending those results to larger classes of graphs seems to be demanding but interesting problem. Finally, considering other irregularity measures could offer new insights in the topic.

References


