Copulas: an Approach How to Model the Dependence Structure of Random Vectors

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Abstract: Copulas enabling to characterize the joint distributions of random vectors by means of the corresponding one-dimensional marginal distributions are presented and discussed. Some properties of copulas and several construction methods, especially when a partial knowledge is available, are included. Possible applications are indicated.

Keywords: random variable, random vector, dependence, copula, aggregation function

1 Introduction

Random vectors are fully characterized by the distribution function. Indeed, if $H = (X_1, \ldots, X_n)$ is an n-dimensional random vector, then all its probabilistic characteristics can be computed by means of the distribution function F_H: $\mathbb{R}^n \to [0, 1]$, $F_H(x_1, \ldots, x_n) = P(X_1 \le x_1 \,\&\, \ldots \,\&\, X_n \le x_n)$. Note that F_H is non-decreasing in any component and it fulfils limit boundary conditions

$$\lim_{t\to-\infty} F_H(x_1,\cdots,x_{i-1},t,x_{i+1},\cdots,x_n) = 0 \text{ , i = 1, …, } n, \tag{1}$$

and

$$\sup\left\{F_H(x_1,\cdots,x_n) \mid (x_1,\cdots,x_n) \in \mathbb{R}^n\right\} = 1. \tag{2}$$

Moreover, F_H is n-increasing (compare (4)). However, F_H can be a rather complicated function and its relationship with the marginal distribution functions $F_{X_1},\cdots,F_{X_n} : \mathbb{R} \to [0, 1]$,

$$F_{X_i}(t) = \sup\left\{F_H(x_1,\cdots,x_{i-1},t,x_{i+1},\cdots,x_n) \mid (x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n) \in \mathbb{R}^{n-1}\right\}$$

is not transparent, in general.

To clarify the above mentioned relationships, after studies of Hoeffding and Fréchet, the notion of a copula was introduced by Sklar [37]. For two-dimensional case, a copula C: $[0, 1]^2 \to [0, 1]$ is an aggregation function with neutral element 1, which is 2-increasing, i.e.,

$$C(x, y) + C(x', y') - C(x, y') - C(x', y) \geq 0 \tag{3}$$

for all $x \leq x', y \leq y'$.

Similarly, copulas with higher dimension n can be introduced, replacing 2-increasingness (3) by n-increasingness:

$$\sum_{k \in \{-1,1\}^n} \left(\prod_{i=1}^{n} k_i \right) C\left(x_1^{(k_1)}, \cdots, x_n^{(k_n)}\right) \geq 0, \tag{4}$$

for all $0 \leq x_1^{(-1)} \leq x_1^{(1)} \leq 1, \cdots, x_n^{(-1)} \leq x_n^{(1)} \leq 1$. Note that 1 is a weak neutral element of C, $C(x_1, ..., x_n) = x_i$ whenever $x_j = 1$ for all j ≠ i and 0 is an annihilator of C, $C(x_1, \ldots, x_n) = 0$ whenever $0 \in \{x_1, \ldots, x_n\}$. As an example recall the copula Π: $[0, 1]^n \to [0, 1]$, $\Pi(x_1, ..., x_n) = x_1 \ldots x_n$ which is an n-copula for each $n \geq 2$. When speaking about copulas without specification of their dimension, we will always have in mind 2-copulas, i.e., 2-dimensional case.

According to Sklar's theorem, for any random vector $H = (X, Y)$ there is a copula C such that

$$F_H(u,v) = C(F_X(u), F_Y(v)) \quad \text{for all } u, v \in \mathbb{R}$$

where F_H, F_X, F_Y are the distribution functions of H, X, Y, respectively. Moreover, C is determined uniquely on Ran F_X x Ran F_Y (more precisely, on $\overline{\text{Ran} F_X}$ x $\overline{\text{Ran} F_Y}$), and the restriction $C|_{(\text{Ran } F_X \text{ x Ran } F_Y)}$ is called a subcopula. Vice-versa, for any closed subsets A, B of [0, 1] containing 0 and 1, a mapping D: A x $B \to [0, 1]$ which is 2-increasing, with neutral element 1 and which is also non-decreasing, is always a subcopula of some copula C. Evidently, if $H = (X, Y)$ is a discrete random vector, the corresponding subcopula, which is then unique, is defined on a discrete set.

Example 1. Let V, W, Z be independent random variables uniformly distributed over [0, 1], and let $X = max(V, Z)$, $Y = max(W, Z)$. For the random vector $H = (X, Y)$, its joint distribution function F_H: $\mathbb{R}^2 \to [0, 1]$ is given by $F_H(x, y) = xy\ min(x, y)$, and the marginal distribution function F_X, F_Y on [0, 1] are given by $F_X(x) = F_Y(x) = x^2$. The copula C: $[0, 1]^2 \to [0, 1]$ linking X and Y is given by $C(x, y) = \sqrt{xy\ min(x,y)}$ and it is so called Cuadras – Augé copula with parameter 0.5 (geometric mean of product and *min*).

Similarly, if $H = (X_1, \ldots, X_n)$ there is an n-dimensional copula C: $[0, 1]^n \to [0, 1]$ so that $F_H(x_1, \ldots, x_n) = C(F_{X_1}(x_1), \cdots, F_{X_n}(x_n))$. For more details about copulas we recommend Nelsen's book [30] and the monograph [35].

The aim of this paper is to bring a short description of basic properties of copulas, of some classes of copulas, and especially of some new construction methods for copulas. Though mostly we will deal with 2-copulas, also some examples and results for n-copulas will be included. Finally, we recall some quantitative characteristics of copulas and their applications.

2 Binary Copulas

For any 2-copula C it holds $W \le C \le M$, where $W(x, y) = max\,(x + y - 1, 0)$ and $M(x, y) = min\,(x, y)$, and both W and M are 2-copulas. Moreover, the class of all 2-copulas is convex. Copulas are 1-Lipschitz and thus continuous aggregation functions. If they are, moreover, associative (as binary functions), they are t-norms [16], and thus they have representation as M-ordinal sums of Archimedean copulas. Due to Moynihan [29], an Archimedean copula C: $[0, 1]^2 \to [0, 1]$, i.e., an associative copula satisfying $C(x, x) < x$ for each $x \in\,]0,1[$, is representable in the form

$$C(x, y) = t^{-1}(min(t(0), t(x) + t(y))), \tag{5}$$

where $t : [0,1] \to [0,\infty]$ is a strictly decreasing convex continuous function with $t(1)=0$ (note that the opposite is also true). As an example, take t: $[0,1] \to [0, \infty]$ given by $t(x) = \frac{1-x}{x}$. Then, for $(x, y) \ne (0, 0)$,

$$C(x, y) = \frac{x\,y}{x + y - x\,y}. \tag{6}$$

Note that this copula is called Ali-Mikhail-Haq copula. Observe that the product copula Π is generated by t_Π, $t_\Pi(x) = -log\,x$, while W is generated by t_W, $t_W(x) = 1 - x$.

Coming back to the situation that a copula C models the dependence structure of a continuous random vector (X, Y), note that $C = \Pi$ if and only if X and Y are independent. Moreover, $C = M$ means the total positive dependence of X and Y, i.e., $Y = f(X)$ for some function f strictly increasing on Ran X. Similarly, $C = W$ means the total negative dependence of X and Y, i.e., $Y = f(X)$ for some function f strictly decreasing on Ran X.

Suppose that random variables X and Y are coupled by a copula C. Then for any increasing $\mathbb{R} \to \mathbb{R}$ transformations f_1, f_2 and any decreasing $\mathbb{R} \to \mathbb{R}$ transformations g_1, g_2, random variables $f_1(X)$ and $f_2(Y)$ are also coupled by C, but random variables $f_1(X)$ and $g_2(Y)$ are coupled by a copula $C^-: [0, 1]^2 \to [0, 1]$ given by

$$C^-(x, y) = x - C(x, 1 - y), \tag{7}$$

and similarly, random variables $g_1(X)$ and $f_2(Y)$ are coupled be a copula $C_-: [0, 1]^2 \to [0, 1]$ given by

$$C_-(x, y) = y - C(1 - x, y), \tag{8}$$

Random variables $g_1(X)$ and $g_2(Y)$ are coupled by the survival copula $\hat{C}: [0, 1]^2 \to [0, 1]$ given by

$$\hat{C}(x, y) = x + y - 1 + C(1 - x, 1 - y). \tag{9}$$

Observe that

$$\left(C^-\right)^- = \left(C_-\right)_- = \left(\hat{C}\right)^{\wedge} = C$$

and

$$\left(C^-\right)_- = \left(C_-\right)^- = \hat{C}.$$

Several interesting results concerning these related copulas can be found in [17]. Note that constructions (7), (8), (9) allow to extend or modify several results for copulas. As a typical example recall W-ordinal sums introduced in [27]. This new type of ordinal sums for copulas can be derived from the standard ordinal sums of copulas (we will call them M-ordinal sums) by means of either (7) or (8). Indeed, for a disjoint system $\left(]a_\alpha, b_\alpha[\right)_{\alpha\in A}$ of open subintervals of [0,1], $C = W\left(\langle a_\alpha, b_\alpha, C_\alpha\rangle \mid \alpha \in A\right)$ is given by

$$C(x,y) = \begin{cases} (b_\alpha - a_\alpha)C_\alpha\left(\frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - (1 - b_\alpha)}{b_\alpha - a_\alpha}\right) & \text{if } (x, y) \in [a_\alpha, b_\alpha] \times [1\text{-}b_\alpha, 1\text{-}a_\alpha] \\ W(x,y) & \text{else,} \end{cases} \tag{10}$$

if and only if

$$C^- = M\left(\langle a_\alpha, b_\alpha, C_\alpha^-\rangle \mid \alpha \in A\right),$$

i.e.,

$$C^{-}(x,y)=\begin{cases} a_\alpha+(b_\alpha-a_\alpha)C_\alpha^{-}\left(\dfrac{x-a_\alpha}{b_\alpha-a_\alpha},\dfrac{y-a_\alpha}{b_\alpha-a_\alpha}\right) & \text{if } (x,y)\in[a_\alpha,b_\alpha]^2 \\ M(x,y) & \text{else.}\end{cases} \tag{11}$$

Another type of ordinal sums based on a gluing method from [36] are so called horizontal (or vertical) g-ordinal sums. Following [27],

$C = hg - \left(\langle a_\alpha, b_\alpha, C_\alpha\rangle \mid \alpha \in A\right)$, is given by

$$C(x,y)=\begin{cases} a_\alpha+(b_\alpha-a_\alpha)C_\alpha\left(\dfrac{x-a_\alpha}{b_\alpha-a_\alpha},y\right) & \text{if } x\in\,]a_\alpha,b_\alpha[\\ \Pi(x,y) & \text{else,}\end{cases}$$

where $\{]a_\alpha, b_\alpha[\}_{\alpha\in A}$ is disjoint system of open subintervals of [0, 1] and $(C_\alpha)_{\alpha\in A}$ is a system of 2-copulas.

Note that all types of ordinal sums are not only construction methods, but also representation tools.

An interesting construction method for 2-copulas was introduced in [3]. Let C_1, C_2 be two 2-copulas. Then their product $C = C_1 * C_2$ given by

$$C(x,y)=\int_0^1 \frac{\partial C_1(x,t)}{\partial t}.\frac{\partial C_2(t,y)}{\partial t}dt \tag{12}$$

is also a 2-copula. Note that the product $*$ is an associative operation on the set of all 2-copulas with neutral element M and annihilator Π.

An important generalization of the product $*$ in (12) was recently proposed in [5] and is based on a family $\Delta = (D_t)_{t\in[0,1]}$ of copulas,

$$C_1 *_\Delta C_2(x,y)=\int_0^1 D_t\left(\frac{\partial C_1(x,t)}{\partial t},\frac{\partial C_2(t,y)}{\partial t}\right)dt\,. \tag{13}$$

Evidently, if $D_t = \Pi$ for each $t \in [0,1]$, (12) and (13) coincide. As examples, observe that $C *_\Delta W = C^{-}$, see (7), and $W *_\Delta C = C\_$, see (8).

We also recall a new method for constructing copulas introduced by Mayor, Mesiar and Torrens in [22].

Proposition 1. Let φ: [0,1] → [0,1] be a convex function, such that for each $x \in [0, 1]$, $\varphi(x) \geq 2x - 1$, $\varphi(0) = 0$. Then the function $C_{(\varphi)}$: $[0, 1]^2 \to [0, 1]$ given, for $(x, y) \neq (0, 0)$, by

$$C_{(\varphi)}(x,y) = \varphi\left(max(x,y)\varphi^{(-1)}\left(\frac{min(x,y)}{max(x,y)}\right)\right), \tag{14}$$

is a copula. Here $\varphi^{(-1)}$: [0, 1] → [0, 1] is the pseudo-inverse of φ given by

$\varphi^{(-1)}(x) = sup\ \{t \in [0,1] \mid \varphi(t) \le x\}$.

Note that $C_{(\varphi)}$ is symmetric and $C_{(\varphi)}(x, x) = \varphi(x)$, i.e., φ is the diagonal section of $C_{(\varphi)}$.

Let for $c \in [0, 1]$, φ_c: [0, 1] → [0, 1] be given by $\varphi_c(x) = max(c\,x, 2\,x - 1)$. Then Proposition 1 can be applied, and we can introduce a parametric class of copulas $\left(C_{(\varphi_c)}\right)_{c\in[0,1]}$,

$$C_{(\varphi_c)}(x,y) = \begin{cases} y & \text{if } (2-c)\,y \le c\,x \\ x & \text{if } (2-c)\,x \le c\,y \\ x+y-1 & \text{if } x+y \ge \dfrac{2}{2-c} \\ \dfrac{c(x+y)}{2} & \text{else.} \end{cases} \tag{15}$$

For any given 2-copula C, we have introduced in [27] a method yielding a parametric system of copulas $\left(C_{(\alpha)}\right)_{\alpha\in]0,1[}$. Based on the idea of conditional distribution functions, for $\alpha \in \,]0,1[$, $C_{(\alpha)} : [0,1]^2 \to [0,1]$ is given by

$$C_{(\alpha)}(x,y) = \frac{C(sup\{\,t \in [0,1] \mid C(t,\alpha) \le \alpha\,x\},\, \alpha\,y)}{\alpha}.$$

Take, for example Ali-Mikhail-Haq copula C given by (6). Then

$$C_{(\alpha)}(x,y) = \frac{C\left(sup\left\{\ t \in [0,1] \mid \dfrac{t\alpha}{t+\alpha-t\alpha} \le \alpha\,x\right\}, \alpha\,y\right)}{\alpha} = \frac{C\left(\dfrac{x\alpha}{1-x(1-\alpha)}, y\right)}{\alpha} =$$

$$= \frac{x\,y}{x\ +\ y - x\,y} = C(x,y),$$

i.e., Ali-Mikhail-Haq copula C is invariant under conditioning $C_{(\alpha)}$. On the other hand, starting from the copula $C_{(\varphi_c)}$, given in (15), we have $\left(C_{(\varphi_c)}\right)_{(\alpha)} \ne \left(C_{(\varphi_c)}\right)_{(\beta)}$, whenever α, β ∈ [c, 1[and α ≠ β, but $\left(C_{(\varphi_c)}\right)_{(\alpha)} \ne \left(C_{(\varphi_c)}\right)_{(c)}$ for each α ∈]0, c]. Moreover, $\left(C_{(\varphi_c)}\right)_{(c)}$ is given by

$$\left(C_{(\varphi_c)}\right)_{(c)}(x,y)=\begin{cases} x & \text{if } (2-c)\,x\le c\,y \\ \dfrac{c(x+y)}{2} & \text{if } \dfrac{c}{(2-c)}x\le y\le \dfrac{2-c}{c}x\le 1 \\ x+\dfrac{c}{2}(y-1) & \text{if } \dfrac{c}{2-c}\le x\le \dfrac{y+c}{2} \\ y & \text{else.}\end{cases}.$$

3 *n* - copulas

Both W and M are associative 2-copulas and thus they can be univocally extended to n-ary functions. For each n-copula C: $[0, 1]^n \to [0, 1]$ it still holds $W \le C \le M$ (using the same notation for binary and n-ary forms of W and M). While M is copula for each $n \ge 2$, W is copula if and only if $n = 2$. Indeed, for $C = W$, putting $x_i^{(1)} = 1$ and $x_i^{(-1)} = 0.5$, $i = 1, \ldots, n$, the inequality (4) reduces to $1-\frac{n}{2}\ge 0$. Similarly, each associative 2-copula has a genuine extension to an n-ary function. In the case of Archimedean copulas represented by (5), their n-ary form can be written as follows

$$C(x_1,\cdots,x_n)=t^{-1}\left(min\left(t(0),\sum_{i=1}^{n}t_i(x_i)\right)\right). \tag{16}$$

However, formula (16) describes an n-ary copula (for each $n \ge 2$) only if the pseudo-inverse $t^{(-1)}$: $[0, \infty] \to [0, 1]$ given by

$$t^{(-1)}(x)=t^{-1}\left(min\left(t(0),x\right)\right)$$

is absolutely monotone, i.e., it possesses all derivatives on $]0, \infty[$ which alternate the sign (evidently, the first derivative is negative). For more details see [11]. In general, among associative 2-copulas one can extend to n-copulas (for each $n \ge 2$) only M-ordinal sums of Archimedean copulas generated by inverses of absolutely monotone bijections φ: $[0, \infty] \to [0, 1]$. These resulting operations were considered recently in the logical environment by Radojevič [32]. For fixed $n > 2$, formula (16) yields an n-ary copula if and only if the function f: $[-\infty, 0] \to [0, \infty]$ given by $f(x) = t^{(-1)}(-x)$ has non-negative differences of orders 1, 2, …, n. Note that this means that f is $(n - 2)$-times differentiable (and all these derivatives are non-negative), and $f^{(n-2)}$ is a convex function. For more details see [25]. We recall two other construction methods for n-copulas.

Method 1. For any n_i-copulas C_i: $[0,1]^{n_i}\to[0,1]$, i =1, …, k, the function C: $[0,1]^{n_1+\cdots+n_k}\to[0,1]$ given by

$$C(x_1,\cdots,x_{n_1},x_{n_1+1},\cdots,x_{n_1+\cdots+n_k}) = C_1(x_1,\cdots,x_{n_1}).C_2(x_{n_1+1},\cdots,x_{n_1+n_2}).\cdots.$$
$$C_k(x_{n_1+\cdots+x_{n-k}+1},\cdots,x_{n_1+\cdots+n_k})$$

is an n-copula, where $n = \sum_{i=1}^{k} n_i$ (note that if $n_i = 1$ then $C_i(x) = x$ for all $x \in [0, 1]$ by convention).

Method 2. Let $\varphi_1, \ldots, \varphi_n$: $[0, 1] \to [0, 1]$ be Lebesgue measure λ preserving functions, i.e., for each Borel subset $E \subset [0, 1]$, $\lambda(\varphi_i^{-1}(E)) = \lambda(E)$, i = 1, …, n. Then the function given by

$$C(x_1,\cdots,x_n) = \lambda\left(\varphi_1^{-1}([0,x_1]) \cap \cdots \cap \varphi_n^{-1}([0,x_n])\right) \tag{17}$$

is an n-copula.

Note that each n-copula can be represented in the form (17), for more details see [20].

Observe that if $\varphi_1 = \ldots = \varphi_n$ then $C = M$ is the strongest n-copula. If, for example, $\varphi_1 = \varphi_2 = 1 - \varphi_3 = id_{[0,1]}$, then (17) results in a 3-copula C: $[0, 1]^3 \to [0, 1]$ given by

$$C(x,y,z) = \lambda([0,x] \cap [0,y] \cap [1-z,1]) = max(min(x,y) + z - 1, 0) = W(M(x,y),z).$$

4 Discrete Copulas

An interesting class of subcopulas are discrete copulas introduced in [19], compare also empirical copulas discussed in [30], D: $I_n^2 \to [0,1]$ (or in m–dimensional case, D: $I_n^m \to [0,1]$), where $I_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\right\}$, and irreducible discrete copulas K: $I_n^2 \to I_n$ (K: $I_n^m \to I_n$) introduced in [22] (in an equivalent form on the scales $L_n = \{0, 1, \ldots, n\}$), see also [24].

Discrete copulas D: $I_n^2 \to [0,1]$ are in a one-to-one correspondence with bistochastic n x n matrices [19]. Several properties and constructions for discrete copulas can be thus introduced by means of properties, notions and constructions of bistochastic matrices. For example, the product of copulas $C_1 * C_2$ mentioned in the previous section, see (12), has its discrete counterpart $D_1 * D_2$ described by the product of the corresponding bistochastic matrices. The class $\mathcal{D}_n$ of all discrete $I_n^2 \to [0,1]$ copulas is a polyhedron with vertices corresponding to the permutation matrices of order n. However, then the corresponding irreducible discrete copulas are just discrete copulas with range I_n, as introduced in [22]. Each such copula related to a permutation σ describes the ordered statistics of x and y samples which are coupled together, i.e., if x_i is the j'th order statistics in the x sample, then y_i is the $\sigma(j)$'th order statistics in the y sample. For more details we recommend [26]. Similarly, m-dimensional case for $m > 2$ can be treated. Indeed, D: $I_n^m \to I_n$ is a discrete copula if and only if there are permutations $\sigma_1, \sigma_2, \ldots, \sigma_m$ of $(1, 2, \ldots, n)$ such that the sample

$$(x_{11},\cdots,x_{1m}),\cdots,(x_{n1},\cdots,x_{nm}),$$

with distinct values on each fixed coordinate, can be written in the form

$$\left(x'_{1\sigma_1(1)},\cdots,x'_{1\sigma_m(1)}\right),\cdots,\left(x'_{n\sigma_1(n)},\cdots,x'_{n\sigma_m(n)}\right),$$

where x'_{ij} is the j^{th} order statistics in the sample from i^{th} coordinate.

5 Copulas Based on a Partial Knowledge

Partial knowledge about the relationship of random variables X and Y restricts the choice of a copula C coupling X and Y. In several cases, such knowledge determines the values of C on a subset of domain $[0, 1]^2$ only, and we want to extend this information to the whole domain of C. Rarely such extension is unique, and thus we mostly look for some extremal (or simple) extensions. A typical case is when knowing the diagonal section $\delta: [0, 1] \to [0, 1]$ of a copula C, $\delta(x) = C(x, x)$, i.e., for [0, 1] uniformly distributed random variables X and Y, knowing the distribution function of $Z = max(X, Y)$. There always exists a copula whose diagonal section coincides with given δ, so-called diagonal copula [8], [30], given by

$$C_\delta(x,y) = min\left(x,\ y,\ \frac{\delta(x)+\delta(y)}{2}\right).$$

Moreover, there is always a weakest copula $B_\delta : [0,1]^2 \to [0,1]$ with $B_\delta(x,x) = \delta(x),\quad x \in [0,1]$. The copula B_δ is given by

$$B_\delta(x,y) = min(x,y) - \min_{t\in[x\wedge y,\, x\vee y]} (t - \delta(t)),$$

and called the Bertino copula, see [2], compare also [9], [13].

In general, the strongest copula with given diagonal section δ need not exist. Observe that C_δ is always a maximal element of the class of copulas with diagonal section δ, and it is the strongest symmetric copula of that class.

The problem when a function MT_δ: $[0, 1]^2 \to [0, 1]$,

$$MT_\delta(x,y) = max(0, \delta(x \vee y) - |x - y|)$$

is a copula was solved in [5]. The function MT_δ is a copula (so-called Mayor-Torrens copula) if and only if the function $\delta - id$ is non-decreasing on $\delta^{-1}(]0, 1])$. Note that then $MT_\delta = B_\delta$ and

$$MT_\delta(x,y) = min(x,y) - min(x - \delta(x), y - \delta(y)).$$

Similar results were studied in the case of given opposite diagonal section ω: $[0, 1] \to [0, 1]$, $\omega(x) = C(x, 1-x)$, see [12], [13]. The function

$$C_\omega(x,y) = max\left(0, x + y - 1, \frac{\omega(x) + \omega(y)}{2}\right)$$

is always a copula with opposite diagonal section ω. There is always a strongest copula with given ω, see [12], [13]. Observe that the above results were straightforwardly shown in [12], [13], however, they can be derived from results for diagonal sections exploiting the constructions (7) or (8), see [14].

Recent results concerning the extensions from affine sections of copulas can be found in [14]. Moreover, in [15] we have discussed extensions of horizontal sections of copulas, h: $[0, 1] \to [0, 1]$, $h(x) = C(x, b)$ for a fixed $b \in]0, 1[$. For example, the function C_h: $[0, 1]^2 \to [0, 1]$ given by

$$C_h(x,y) = \begin{cases} \dfrac{h(x)y}{b} & \text{if } y \le b \\ \dfrac{h(x)(1-y) + x(y-b)}{1-b} & \text{else} \end{cases},$$

is always a copula with prescribed horizontal section h. Note that related results for irreducible discrete copulas were discussed in [21].

6 Some Quantitative Characteristics of Binary Copulas

Among several quantitative characteristics of 2-copulas we introduce two of them.

Kendall's tau $\tau_{X,Y}$ characterizes a random vector (X,Y) in the next sense: if (x_1, y_1), …, (x_n, y_n) is a statistical sample describing (X,Y), then the estimation of $\tau_{X,Y}$ is given by

$$\hat{\tau}_{X,Y} = \frac{c-d}{\binom{n}{2}},$$

where c is the number of concordant pairs (x_i, y_i) and (x_j, y_j) (i.e., such that $(x_i-x_j)(y_i-y_j) > 0$) and d is the number of discordant pairs (i.e., $(x_i-x_j)(y_i-y_j) < 0$). The population Kendall's tau can be computed by means of the copula C linking X and Y,

$$\tau_{X,Y} = 4\iint_{[0,1]^2} C(x,y)\,d\,C(x,y) - 1 .$$

Another quantitative characteristic of dependence of random variables X and Y is Spearman's rho. For a sample (x_1, y_1), …, (x_n, y_n), $\hat{\rho}_{X,Y}$ is the rank correlation coefficient. Population $\rho_{X,Y}$ can be computed by formula

$$\rho_{X,Y} = 12\iint_{[0,1]^2} C(x,y)\,dx\,dy - 3 .$$

Note that both, $\tau_{X,Y}$, $\rho_{X,Y} \in [-1, 1]$ and $\tau_{X,Y} = \rho_{X,Y} = 1$ if and only if $C = M$, i.e., if $Y = f(X)$ for some function f strictly increasing on Ran X. Similarly, $\tau_{X,Y} = \rho_{X,Y} = -1$ means that $C = W$ and $Y = f(X)$ for some function f strictly decreasing on Ran X. Thus, both $\tau_{X,Y}$ and $\rho_{X,Y}$ are indicators of monotone functional dependence of X and Y. Recall that Pearson's rho (the standard correlation coefficient) describes the degree of linear dependence of X and Y, while Spearman's rho describes the rank correlation coefficient. This fact allows to apply copulas to check the monotone functional dependence of random variables X and Y.

7 Some Applications

Risk management in financial or hydrological environment is based on the conditional behavior of extremal events, expressed by the quantity (if it exists)

$$UT_{X,Y} = \lim_{\alpha\to 0^+} P\left(X \geq Q_{X,1-\alpha} \mid Y \geq Q_{Y,1-\alpha}\right) \tag{18}$$

where $Q_{X,\alpha}$ is the α-quantile of random variable X. $UT_{X,Y}$ is called the upper tail dependence. Similarly, the lower tail dependence $LT_{X,Y}$ can be defined, when in (18) α approaches to 1^-.

If the dependence of X and Y is captured by a copula C, then

$$UT_{X,Y} = 2 - \delta'_C\left(1^-\right) \quad \text{and} \quad LT_{X,Y} = \delta'_C\left(0^+\right),$$

where δ_C: $[0, 1] \to [0, 1]$ is the diagonal section of C.

Example 2. An interesting class of copulas is determined by triangulation method, see [4]. For $(x, y) \in]0, 1[^2$, let $\alpha \in [W(x, y), M(x, y)]$. Then the copula $C_{x, y, \alpha}$: $[0, 1]^2 \to [0, 1]$ which is linear on four triangles determined by the point (x, y) and vertices of the unite square $[0, 1]^2$, is given by

$$C_{x,y,\alpha}(u,v) = \begin{cases} \dfrac{\alpha u}{x} & \text{if } (u,v) \in \Delta_{(0,0),(0,1),(x,y)}, \\ \dfrac{\alpha v}{y} & \text{if } (u,v) \in \Delta_{(0,0),(1,0),(x,y)}, \\ \dfrac{(y-\alpha)(u-1)}{1-x} & \text{if } (u,v) \in \Delta_{(1,0),(1,1),(x,y)}, \\ \dfrac{(x-\alpha)(v-1)}{1-y} & \text{if } (u,v) \in \Delta_{(0,1),(1,1),(x,y)}. \end{cases}$$

Then

$$UT_{C_{x,y,\alpha}} = \frac{1+\alpha-x-y}{1-min(x,y)} \in \left[\frac{max(1-x-y,0)}{1-min(x,y)}, \frac{1-max(x,y)}{1-min(x,y)}\right]$$

and

$$LT_{C_{x,y,\alpha}} = \frac{\alpha}{max(x,y)} \in \left[\frac{max(x+y-1,0)}{max(x,y)}, \frac{min(x,y)}{max(x,y)}\right].$$

Moreover,

$$\rho_{C_{x,y,\alpha}} = 1+4\alpha-2x-2y \in [-1+2|x+y-1|, 1-2|x-y|].$$

For modeling real data (financial, hydrological, sociological, etc.) by means of copulas, there are applied several methods. For a fixed set of copulas (mostly some parametric family of Archimedean copulas) and the observed sample (x_1, y_1), …, (x_n, y_n), we can first compute the empirical parameters $\hat{\tau}$ and/or $\hat{\rho}$ and fit the best copula by means of its Kendall's tau and/or Spearman's rho. Another approach is based on the least square method. There are alternative approaches based on specific characteristics of copulas and their empirical estimations. Among rare methods dealing with asymmetric copulas, i.e., with the case of nonexchangeable random vectors, there are least square method-based approaches related to asymmetric Archimedean copulas or to Archimax copulas. Note that we are just working on a software for fitting such copulas to real data.

Another field of applications of copulas is in the nonadditive integral area supporting the multicriteria decision aid. For a set $X = \{1, \dots, k\}$ of criteria, the capacity m: $2^X \to [0, 1]$ assigns to group of criteria E a weight $m(E)$ (thus $E_1 \subset E_2$ implies $m(E_1) \le m(E_2)$, $m(\emptyset) = 0$, $m(X) = 1$). For a given score vector $\mathbf{x} \in [0, 1]^X$ and a 2-copula C: $[0, 1]^2 \to [0, 1]$, the corresponding C-based integral $I_{C, m}(\mathbf{x})$ can be understood as the utility of $\mathbf{x}$ and it is given by

$$I_{C,m}(\mathrm{x}) = \sum_{i=1}^{k} \Big(C\big(x_{\sigma(i)}, m(E_{\sigma,i})\big) - C\big(x_{\sigma(i)}, m(E_{\sigma,i+1})\big) \Big),$$

where σ is a permutation of (1, …, k) such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(k)}$, and $E_{\sigma,i} = \{\sigma(i), \ldots, \sigma(k)\}$, with convention $E_{\sigma,k+1} = \emptyset$. For more details see [18].

Conclusion

We have discussed binary and n-ary copulas, including some construction methods and applications. Copulas bring a new light into stochastic dependence modeling and they offer a powerful tool for better fitting of models of several real world problems, especially in connection with extremal events. The state-of-art overview of applications of copulas in problems occurring in nature, e.g., hydrological problems, can be found in a recent monograph [34] while several financial applications of copulas are discussed, e.g., in [7], [10].

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References

[1] Alsina, C., Nelsen, R. B., Schweizer, B.: On the Characterization of a Class of Binary Operations on Distributions Functions. *Stat. Probab. Lett*. Vol. **17**, 1993, pp. 85-89

[2] Bertino, S.: On Dissimilarity between Cyclic Permutations. *Metron*, Vol. **35**, 1977, pp. 53-88, in Italien

[3] Darsow, W. F., Nguyen, B., Olsen, E. T.: Copulas and Markov Processes. *Illinois J. Math.*, Vol. **36**, 1992, pp. 600-642

[4] De Baets, B., De Meyer, H., Mesiar, R.: Binary Aggregation Functions Based on Triangulation, submitted

[5] Durante, F., Klement, E. P., Quesada-Molina, J. J.: Copulas, compatibility and Fréchet classes, submitted

[6] Durante, F., Mesiar, R., Sempi, C.: On a Family of Copulas Constructed from the Diagonal Section. *Soft Computing*, Vol. **10**, 2006, pp. 490-494

[7] Embrechts, P., Lindskog, F., McNeil, A. J.: *Modelling Dependence with Copulas and Applications to Risk Management. Handbook of Heavy Tailed Distributions in Finance*. S. T. Rachev, ed. Elsevier, Amsterdam, 2003, pp. 329-384

[8] Fredricks, G. A., Nelsen, R. B.: Copulas Constructed from Diagonal Sections. *Distributions with Given Marginals and Moment Problems*, V. Beneš and J. Štepán, eds. Kluwer Academic Publishers, Dordrecht, 1997, pp. 129-136

[9] Fredricks, G. A., Nelsen, R. B.: The Bertino Family of Copulas. *Distributions with Given Marginals and Statistical Modelling*, C. M. Cuadras et al., eds. Kluwer Academic Publishers, Dordrecht, 2002, pp. 81-91

[10] Frees, E. W., Valdez, E. A.: Understanding Relationships Using Copulas. *North Amer. Act. J.*, Vol. **2**, 1998, pp. 1-25

[11] Kimberling, C. H.: On a Class of Associative Functions. *Publ. Math. Debrecen*, Vol. **20**, 1973, pp. 21-39

[12] Klement, E. P., Kolesárová, A.: 1-Lipschitz Aggregation Operators, Quasi-Copulas and Copulas with Given Diagonals. *Soft Methodology and Random Information Systems*, M. Lopéz-Díaz et al., eds., Springer-Verlag Berlin Heidelberg, 2004, pp. 205-211

[13] Klement, E. P., Kolesárová, A.: Extensions to Copulas and Quasi-Copulas as Special 1-Lipschitz Aggregation Operators. *Kybernetika*, Vol. 41, 2005, pp. 329-348

[14] Klement, E. P., Kolesárová, A.: Intervals of 1-Lipschitz Aggregation Operators, Quasi-Copulas, and Copulas with Given Affine Section. *Monatshefte für Mathematik*, Vol. **152**, 2007, pp. 151-167

[15] Klement, E. P., Kolesárová, A., Mesiar, R., Sempi, C.: Copulas Constructed from Horizontal Sections. *Comm. Statistics. Theory and Methods*, Vol. **36**, 2007, pp. 2901-2911

[16] Klement, E. P., Mesiar, R., Pap, E.: *Triangular Norms*. Kluwer Academic Publishers, Dordrecht, 2000

[17] Klement, E. P., Mesiar, R., Pap, E: Invariant Copulas. *Kybernetika*, Vol. **38**, 2002, pp. 275-285

[18] Klement, E. P., Mesiar, R., Pap, E: A Universal Integral Based on Measures of Level Sets, submitted

[19] Kolesárová, A., Mesiar, R., Mordelová, J., Sempi, C.: Discrete Copulas. *IEEE Trans. on Fuzzy Systems*, Vol. 14, pp. 698-705, 2006

[20] Kolesárová, A., Mesiar, R., Sempi, C.: Measure Preserving Transformations, Copulae and Compatibility. *Mediterranean J. of Mathematics*, accepted

[21] Kolesárová, A., Mordelová, J.: Quasi-Copulas and Copulas on a Discrete Scale. *Soft Computing*, Vol. **10**, 2006, pp. 495-501

[22] Mayor, G., Mesiar, R., Torrens, J.: Quasi-Homogeneous Copulas, *Kybernetika*, Vol. 44 (2008), pp. 745-756

[23] Mayor, G., Suňer, J., Torrens, J.: Copula-like Operations on Finite Settings. *IEEE Trans. On Fuzzy Systems*, Vol. **13**, No. 4, 2005, pp.468-477

[24] Mayor, G., Torrens, J.: Triangular Norms on Discrete Settings. *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms*, E. P. Klement and R. Mesiar, eds., Elsevier, pp. 2005, 189-230

[25] Mc Neil, A. J., Nešlehová, J.: Multivariate Archimedean Copulas, *d*-Monotone Functions and l_1-norm Symmetric Distributions, submitted

[26] Mesiar, R.: Discrete Copulas: What They are. *Proc. EUSFLAT-LFA'2005*, Barcelona, 2005, pp. 927-930

[27] Mesiar, R., Jágr, V., Juráňová, M., Komorníková, M.: Univariate conditioning of copulas. *Kybernetika*, Vol. 44 (2008), pp. 807-816

[28] Mesiar, R., Szolgay, J.: W-Ordinal Sums of Copulas and Quasi-Copulas. *Proc. MAGIA-UWPM'2005*, Publishing House STU, Bratislava, 2005, pp. 78-83

[29] Moynihan, R.: On τ_T Semigroups of Probability Distribution Functions II. *Aequationes Math*., Vol. **17**, 1978, pp. 19-40

[30] Nelsen, R. B.: *An Introduction to Copulas*. Lecture Notes in Statistics 139, Springer Verlag, New York, 1999

[31] Nelsen, R. B., Fredricks, G. A.: Diagonal Copulas. *Distributions with Given Marginals and Moment Problems*, V. Beneš and J. Štepán, eds. Kluwer Academic Publishers, Dordrecht, 1997, pp. 121-127

[32] Radojevič, D. G.: Logical Aggregation Based on Interpolative Realization of Boolean Algebra. *New Dimensions in Fuzzy Logic and Related Technologies. Proc. of the 5th EUSFLAT Conference*, M. Štepnička, V. Novák, U. Bodenhofer, eds. Ostrava, Czech Republik, September 2007, pp. 119-126

[33] Rodríguez-Lallena, J. A., Úbeda-Flores, M.: Compatibility of Three Bivariate Quasi-Copulas: Applications to Copulas. *Soft Methodology and Random Information Systems*, M. Lopéz-Díaz et al., eds., Springer-Verlag Berlin Heidelberg, 2004, pp. 173-180

[34] Salvadori, G., De Michele, C., Kottegoda, N. T., Rosso, R.: *Extremes in Nature. An Approach Using Copulas*. Springer Verlag, 2007

[35] Schweizer, B., Sklar, A.: *Probabilistic Metric Spaces*. North-Holland, New York, 1983

[36] Siburg, K. F., Stoimenov, P. A.: Gluing Copulas. *Communications in Statistics - Theory and Methods*, vol. **37**, 2008, pp. 3124-3134

[37] Sklar, A.: *Fonctions de Répartitionán Dimensions et Leurs Marges*. Publ. Inst. Statist. Univ. Paris, Vol. 8, 1959, pp. 229-231