Tuning of Fractional Controllers Minimising H_2 and H_∞ Norms

Duarte Valério, José Sá da Costa

Department of Mechanical Engineering, Instituto Superior Técnico, Technical University of Lisbon, GCAR, Av. Rovisco Pais, 1049-001 Lisboa, Portugal, e-mail: {dvalerio, sadacosta}@dem.ist.utl.pt

Abstract: H_2 and H_∞ controllers minimise the H_2 or the H_∞ norm of a suitable loop transfer function involving the plant to control and some weighing transfer functions chosen to fulfil performance specifications. In this paper this type of controllers is developed for the case when the plant and / or the weighing transfer functions are of fractional (commensurate) order. Since no analytical results similar to those existing for the integer case have been found, a genetic algorithm is used to minimise the desired norm. An application to temperature control is used to illustrate the method.

Keywords: H_2 and H_∞ controllers, fractional plants, genetic algorithms, temperature control

1 Introduction

Over the last decades, a technique for devising controllers has been developed that consists essentially in minimising the H_2 or the H_∞ norm of the control loop. These two norms have simple interpretations: to put it plainly, the H_2 norm reflects how much a dynamic system amplifies – or attenuates – its input over all frequencies, and the H_∞ norm reflects how much a dynamic system amplifies – or attenuates – its input at the frequency at which the amplification is maximal. This control technique may be applied to both SISO (single-input, single-output) and MIMO (multiple-input, multiple-output) plants, and its results achieve a remarkable robustness (Lublin *et al.*, 1996; Doyle *et al.*, 1989).

The usual method for developing this sort of controllers is based upon a state-space representation of the plant. It cannot, unfortunately, be applied to plants of fractional order – that is to say, to plants that have a dynamic behaviour corresponding to differential equations involving fractional derivatives. State-space representations for such plants do exist (Malti *et al.*, 2003), but, since they involve fractional derivatives of the states, the algorithms developed for integer-order plants (those with a dynamic behaviour corresponding to differential equations involving usual, integer-order derivatives only) cannot be directly transposed to fractional-order plants.

In this paper this problem is addressed by using a numerical minimisation method to perform the minimisation of the H_2 or the H_∞ norm when the plant is of fractional order. The following sections cover the following issues. Section 2 briefly summarizes a few results from the theory of fractional calculus. Section 3 summarizes algorithms for reckoning the H_2 norm of a plant. Section 4 summarizes algorithms for reckoning the H_∞ norm of a plant. Section 5 describes the control loop usually employed together with H_2 and H_∞ controllers. Section 6 describes algorithms to minimise an appropriate norm of that control loop. Section 7 documents an example of application of this control strategy to a fractional order plant. The paper closes with some conclusions.

2 Fractional Order Systems

Fractional calculus is a generalisation of ordinary calculus. Its main idea is to develop a functional operator *D*, associated to an order ν (not restricted to integer numbers), that generalises the usual notions of derivatives (for a positive ν) and integrals (for a negative ν). There are several alternative definitions of operator *D*; the one addressed here is due to the works of Grünwald and Letnikoff. It generalises the usual definition of derivative:

$$Df(x) \overset{\text{def}}{=} \lim_{h\to 0} \frac{f(x)-f(x-h)}{h} \tag{1}$$

It is easily proved by mathematical induction that higher order derivatives are given by

$$D^n f(x) = \lim_{h\to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh)}{h^n}, \quad n \in \mathbb{N} \tag{2}$$

The gamma function, defined as

$$\Gamma(x) \overset{\text{def}}{=} \begin{cases} \int_0^{+\infty} e^{-y} y^{x-1} dy, \text{ if } x \in \mathbb{R}^+ \\ \dfrac{\Gamma(x+n)}{\prod_{i=0}^{n-1}(x+i)}, \; n = -\lfloor x \rfloor, \text{ if } x \in \mathbb{R}^- \setminus \mathbb{Z} \end{cases} \tag{3}$$

has the convenient property of generalising the factorial to all real numbers (with the exception of negative integers, for which it is not defined), since

$$\Gamma(n+1) = n!,\ n \in \mathbb{N} \tag{4}$$

and this property may be used to generalise combinations for non-natural numbers as follows:

$$\binom{a}{b} \overset{\text{def}}{=} \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} \tag{5}$$

As a consequence, (2) gives the same result as

$$D^n f(x) = \lim_{h\to 0} \frac{\sum_{k=0}^{m} (-1)^k \binom{n}{k} f(x-kh)}{h^n}, \quad m,n \in \mathbb{N} \wedge m > n \tag{6}$$

since

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{n!}{k!\infty} = 0 \tag{7}$$

And expression (6) may be generalised for non-natural differentiation orders as follows:

$$D^\nu f(x) = \lim_{\substack{h\to 0 \\ m\to+\infty}} \frac{\sum_{k=0}^{m} (-1)^k \binom{\nu}{k} f(x-kh)}{h^\nu} \tag{8}$$

Two things should be taken into account. First, the upper limit of the summation *m* has to be taken up to infinity because, since ν need not be integer, terms will not be zero from some value of *k* on. Second, when ν is a negative integer, the result should be an integration, and it would be good to know which integration indexes result from using this definition. This question is easy to answer for $\nu = -1$; we will have

$$D^{-1} f(x) = \lim_{\substack{h\to 0 \\ m\to+\infty}} \frac{\sum_{k=0}^{m} (-1)^k \frac{(-1)^k \Gamma(k+1)}{\Gamma(k+1)\Gamma(1)} f(x-kh)}{h^{-1}} = \lim_{\substack{h\to 0 \\ m\to+\infty}} \sum_{k=0}^{m} f(x-kh)h \tag{9}$$

Now this is the Riemann definition of integral ${}_cD_x^{-1} f(x)$ if $h = (x-c)/m$. So, for orders other than –1, we will have

$${}_cD_x^\nu f(x) \overset{\text{def}}{=} \lim_{\substack{h\to 0 \\ mh=x-c}} \frac{\sum_{k=0}^{m} (-1)^k \binom{\nu}{k} f(x-kh)}{h^\nu} \tag{10}$$

This is the Grünwald-Letnikoff definition of fractional derivative. Notice that when ν is a non-integer positive number operator D still needs integration limits; in other words, D is a local operator only when ν is a natural number (the case of usual derivatives). Thorough expositions of the theory of fractional calculus may be found in (Miller and Ross, 1993; Podlubny, 1999; Samko *et al.*, 1993).

The Laplace transform of D follows rules similar to those for integer derivatives and integrals:

$$\mathcal{L}\left[{}_0D_x^\nu f(x)\right] = s^\nu F(s) \tag{11}$$

Zero initial conditions being assumed, systems with a dynamic behaviour described by differential equations involving fractional derivatives give rise to transfer functions with fractional powers of s.

«Fractional» calculus and «fractional» order systems are the usual names though ν may assume irrational values as well – definition (10) handles both cases without distinction. In practice all orders are known with limited precision and all orders may indeed be assumed rational. Fractional transfer functions $G(s)$ dealt with become far more manageable if there is a Q verifying

$$G(s) = \frac{\sum_{k=1}^{n} a_k s^{k/Q}}{\sum_{k=1}^{m} b_k s^{k/Q}}, \quad Q \in \mathbb{N}\,. \tag{12}$$

Such fractional transfer functions are called commensurate. The rational commensurate order is $1/Q$. All transfer functions addressed hereafter are assumed to be like (12). Such transfer functions may be used to model several physical systems in different areas, such as heat transfer, diffusion, behaviour of viscoelastic materials, electrical circuits, and many more (Podlubny, 1999, pp. 243-308).

3 The H_2 Norm

The H_2 norm of a transfer function matrix G is defined as

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \operatorname{tr}\left[G(j\omega)\overline{G(j\omega)^T}\right] d\omega}\,. \tag{13}$$

Now let $\mathbf{A}$ be a matrix with m lines and n columns. Then the product $\mathbf{A}\overline{\mathbf{A}^T}$ is a square matrix with m lines and m columns. Its elements are given by

$$\left[\mathbf{A}\overline{\mathbf{A}^T}\right]_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik} \overline{\left[\mathbf{A}^T\right]_{kj}} = \sum_{k=1}^{n} \mathbf{A}_{ik} \overline{\mathbf{A}_{jk}} \quad (14)$$

Elements in the main diagonal are given by

$$\begin{aligned} &\left[\mathbf{A}\overline{\mathbf{A}^T}\right]_{ii} = \\ &= \sum_{k=1}^{n} \mathbf{A}_{ik} \overline{\mathbf{A}_{ik}} = \\ &= \sum_{k=1}^{n} \left(\mathrm{Re}\left[\mathbf{A}_{ik}\right] + j\,\mathrm{Im}\left[\mathbf{A}_{ik}\right]\right)\left(\mathrm{Re}\left[\mathbf{A}_{ik}\right] - j\,\mathrm{Im}\left[\mathbf{A}_{ik}\right]\right) = \\ &= \sum_{k=1}^{n} \mathrm{Re}^2\left[\mathbf{A}_{ik}\right] + \mathrm{Im}^2\left[\mathbf{A}_{ik}\right] = \sum_{k=1}^{n} \left|\mathbf{A}_{ik}\right|^2 \end{aligned} \quad (15)$$

From the definition of trace, we get

$$\mathrm{tr}\left[\mathbf{A}\overline{\mathbf{A}^T}\right] = \sum_{i=1}^{m} \sum_{j=1}^{n} \left|\mathbf{A}_{ij}\right|^2 \quad (16)$$

The result above means that, for a MIMO system with *n* lines and *m* columns,

$$\|G\|_2 = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} \|G_{ij}\|_2^2}\,. \quad (17)$$

So the problem of finding the H_2 norm is reduced to SISO systems.

3.1 Integer Plants

Let *G* be an integer order system given by the state-space representation

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}u \\ y &= \mathbf{C}x + \mathbf{D}u \end{aligned} \quad (18)$$

Then its H_2 norm may be found solving one of the equations (Lublin *et al.*, 1996, p. 636)

$$\mathbf{A}\mathbf{L}_C + \mathbf{L}_C\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0} \quad (19)$$

$$\mathbf{A}^T\mathbf{L}_O + \mathbf{L}_O\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0} \quad (20)$$

for $\mathbf{L}_C$ (the controllability gramian matrix) or $\mathbf{L}_O$ (the observability gramian matrix). (These two linear matrix equations belong to a class of equations known as Riccatti type equations.) Then

$$\|\mathbf{G}\|_2 = \sqrt{\mathrm{tr}\left(\mathbf{C}\mathbf{L}_C\mathbf{C}^T\right)} \quad (21)$$

$$\|\mathbf{G}\|_2 = \sqrt{\mathrm{tr}\left(\mathbf{B}^T \mathbf{L}_C \mathbf{B}\right)} \tag{22}$$

3.2 Fractional Plants

No results similar to (21) or (22) have been found for fractional systems, but the norm may be found as follows (Malti *et al.*, 2003). Let $1 / Q$ be the commensurate order of the system, and

$$\left.\begin{array}{l} q = C\ (Q) \\ p = Q - q \end{array}\right\} \Rightarrow p + q = Q \tag{23}$$

$$x = \omega^{1/Q} \Leftrightarrow \omega = x^Q \Rightarrow d\omega = Qx^{Q-1}dx \tag{24}$$

Since the complex conjugate may be obtained changing the sign of the imaginary part, we will have

$$\|G\|_2^2 = \frac{1}{\pi}\int_0^{+\infty} Qx^{Q-1} G(jx) G(-jx)\, dx \tag{25}$$

Let A and B be the polynomials in the numerator and denominator of product

$$G(jx)G(-jx) = \frac{A(x)}{B(x)} \tag{26}$$

Notice that the imaginary unit j has been considered as part of polynomials A and B. If we now let

$$\frac{x^q A(x)}{B(x)} = \frac{R(x)}{B(x)} + \sum_{k=0}^{q+\deg(A)-\deg(B)} a_k x^k \tag{27}$$

(where deg(P) represents the degree of polynomial P), then (25) becomes

$$\|G\|_2^2 = \frac{Q}{\pi}\int_0^{+\infty} x^{p-1} x^q \frac{A(x)}{B(x)} dx = \frac{Q}{\pi}\int_0^{+\infty} x^{p-1} \left(\frac{R(x)}{B(x)} + \sum_{k=0}^{q+\deg(A)-\deg(B)} a_k x^k \right) dx \tag{28}$$

Three cases are to be distinguished when reckoning (28).

3.2.1 Case $q + \deg(A) - \deg(B) > 0$

In this case the summation in (28) is not identically zero. Its integral will be

$$\int_0^{+\infty} \sum_{k=0}^{q+\deg(A)-\deg(B)} a_k x^{k+p-1} dx = \sum_{k=0}^{q+\deg(A)-\deg(B)} \left[\frac{a_k}{k+p} x^{k+p} \right]_0^{+\infty} \tag{29}$$

and since k is 1 or higher and p is positive,

$$x^{k+p}\Big|_{x=+\infty} = +\infty \Rightarrow \|G\|_2 = \infty \tag{30}$$

3.2.2 Case $q + \deg(A) - \deg(B) \le 0 \wedge p \ne 0$

In this case let $B(x)$ have b different poles, $s_1, s_2, \dots s_b$, and let m_k be the multiplicity of pole s_k. Then we may perform a partial fraction expansion of

$$\frac{x^q A(x)}{B(x)} = \sum_{k=1}^{b}\sum_{n=1}^{m_k} \frac{a_{k,n}}{\left(x+s_k\right)^n} \tag{31}$$

(Recall that a pole of multiplicity m will appear m times in the expansion.) Then (28) becomes

$$\|G\|_2^2 = \frac{Q}{\pi}\int_0^{+\infty} \sum_{k=1}^{b}\sum_{n=1}^{m_k} \frac{a_{k,n}x^{p-1}}{\left(x+s_k\right)^n}\,dx = \frac{Q}{\pi}\sum_{k=1}^{b}\sum_{n=1}^{m_k} \frac{a_{k,n}}{s_k^n}\int_0^{+\infty} \frac{x^{p-1}}{\left(\frac{x}{s_k}+1\right)^n}\,dx \tag{32}$$

This becomes (Gradshteyn and Ryzhik, 1980, p. 285)

$$\begin{aligned} \|G\|_2^2 &= \\ &= \frac{Q}{\pi}\sum_{k=1}^{b}\sum_{n=1}^{m_k} \frac{a_k}{s_k^n}(-1)^{n-1}\frac{\pi}{s_k^{-p}}\binom{p-1}{n-1}\operatorname{cosec}(p\pi) = \\ &= \sum_{k=1}^{b}\sum_{n=1}^{m_k} \frac{(-1)^{n-1} Q a_k s_k^{p-n}\binom{p-1}{n-1}}{\sin(p\pi)} \end{aligned} \tag{33}$$

3.2.3 Case $q + \deg(A) - \deg(B) \le 0 \wedge p = 0$

In this case the integration rule quoted above cannot be applied, so an alternative expansion is carried out. Let s_1 be one of the poles of (31), arbitrarily chosen. (Actually it would be better to choose the one minimising numerical errors, but it is difficult to know beforehand which one does.) Then we may write

$$\frac{x^{q-1}A(x)}{B(x)} = \sum_{k=2}^{b} \frac{c_k}{\left(x+s_1\right)\left(x+s_k\right)} + \sum_{k=1}^{b}\sum_{n=2}^{m_k} \frac{d_{k,n}}{\left(x+s_k\right)^n} \tag{34}$$

Notice that poles with multiplicity 1 do not appear in the second summation. After some straightforward calculus, expression (28) becomes

$$\begin{aligned}\|G\|_2^2 &= \frac{Q}{\pi}\left(\sum_{k=2}^{b}\int_0^{+\infty}\frac{c_k}{(x+s_1)(x+s_k)}dx+\sum_{k=1}^{b}\sum_{n=2}^{m_k}\int_0^{+\infty}\frac{d_{k,n}}{(x+s_k)^n}dx\right)=\\ &=\sum_{k=2}^{b}\left[\frac{Q}{\pi}\int_0^{+\infty}\left(\frac{1}{x+s_1}+\frac{-1}{x+s_k}\right)\frac{c_k}{s_k-s_1}dx\right]+\sum_{k=1}^{b}\sum_{n=2}^{m_k}\left[\frac{Q}{\pi}\int_0^{+\infty}\frac{d_{k,n}}{(x+s_k)^n}dx\right]=\\ &=\sum_{k=2}^{b}\left\{\frac{Qc_k}{\pi(s_k-s_1)}\left[\ln\frac{x+s_1}{x+s_k}\right]_0^{+\infty}\right\}+\sum_{k=1}^{b}\sum_{n=2}^{m_k}\left\{\frac{Qd_{k,n}}{\pi(-n+1)}\left[(x+s_k)^{-n+1}\right]_0^{+\infty}\right\}=\\ &=\sum_{k=2}^{b}\left[\frac{Qc_k}{\pi(s_k-s_1)}\ln\frac{s_k}{s_1}\right]+\sum_{k=1}^{b}\sum_{n=2}^{m_k}\frac{Qd_{k,n}s_k^{1-n}}{\pi(-n+1)}\end{aligned} \tag{35}$$

3.3 Summing up

The algorithm for finding the H_2 norm of G may be summed up as follows:

- If G is of integer order ($Q = 1$), then apply (19) and (21) or (20) and (22).
- If G is of fractional order ($Q \neq 1$), then:
  - If G is MIMO, find the H_2 norm of its components and then apply (17).
  - If G is SISO, then:
    - If $q+\deg(A)-\deg(B)>0$, the norm is ∞.
    - If $q+\deg(A)-\deg(B)\leq 0$, then:
      - If $p \neq 0$, apply (33).
      - If $p = 0$, apply (35).

It should be noticed that, even though several different formulas are to be applied depending on the value of Q, the norm is a continuous function thereof.

4 The H_∞ Norm

The H_∞ norm of a transfer function matrix G is defined as

$$\|G\|_\infty = \sup_\omega \max \sigma\left[G(j\omega)\right] \tag{36}$$

where $\sigma(\mathbf{A})$ represents the set of singular values of matrix **A**. (This set has a finite number of values, and thus has a maximum; on the other hand, the set resulting of sweeping all frequencies may have no maximum, but only a supreme value.) If G is SISO, (36) becomes simply

$$\|G\|_{\infty} = \sup_{\omega} \max \left| G(j\omega) \right| . \tag{37}$$

This norm may be found by direct evaluation at several frequencies. Frequencies clearly above or below all the frequencies of poles and zeros need not be searched. The result is, of course, equal to or below the exact result – it can never be above.

5 The Control Paradigm

Roughly speaking, the idea of H_2 and H_∞ controllers is to minimise (at least over a certain range of frequencies we are interested in) one of those norms, ensuring that the input is never amplified to such an extent that instability will arise. It is usual to include judiciously chosen shaping transfer functions in the control loop so that control efforts be exerted at those frequencies desired by the control designer. Should the weights be adequately chosen, it is possible to find, by minimising one of the two norms, a control-loop that is stable and robust to plant variations. These are expected to cause a worse performance but not instability. (The above is of course an oversimplified description; see for instance Lublin *et al.* (1996) or Doyle *et al.* (1989) for details.)

H_2 and H_∞ controllers make use of the control structures of the block diagrams in Figure 1, where **K** is the controller, **A**, **B**, **C** and **D** are the matrixes of the state space representation of **G**, and **L** models how noise affects the states (or the inputs, should **L** = **B**). Vector w collects all inputs (references, noise, disturbances…) save the control actions u. Vector z collects all variables showing the performance of the control system, namely outputs and control actions (whose magnitude may have to be limited). Weights W_1 to W_4 may be transfer functions, and usually are. They let the control designer shape the result by telling the loop in what frequencies control actions, outputs, etc., have to be large or small.

The above interconnections give rise to this transfer function

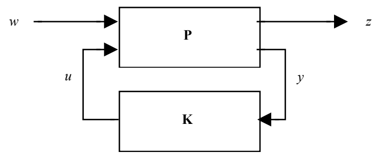
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_3\mathbf{S}\mathbf{G}_1W_1 & W_3\mathbf{S}\mathbf{G}_2\mathbf{K}W_2 \\ W_4\mathbf{K}\mathbf{S}\mathbf{G}_1W_1 & W_4\mathbf{K}\mathbf{S}W_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{38}$$

$$\mathbf{G}_1 = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{L} \tag{39}$$

$$\mathbf{G}_2 = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{40}$$

$$\mathbf{S} = (\mathbf{I} - \mathbf{G}_2\mathbf{K})^{-1} \tag{41}$$

It is a norm of this matrix transfer function that we want to minimise.



where P is given by

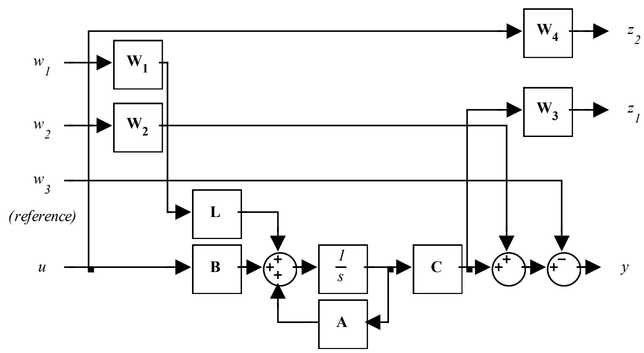


Figure 1
Block diagrams for H_2 and H_∞ controllers

6 Finding Controllers

For integer order plants it is possible to find a controller minimising either the H_2 or the H_∞ norm of a suitable loop transfer function by analytical means (Lublin *et al.*, 1996; Doyle *et al.*, 1989). Unfortunately no such relations have yet been found for fractional-order plants; those for integer-order plants are derived from mathematical formulations for the norms very different from the formulations available for the fractional case. But it is possible to use numerical methods to minimise such norms. (Petras and Hypiusova (2002) also suggest a numerical multicriteria optimisation method for the H_∞ case.) Among all possible optimisation algorithms, genetic algorithms have been used in this paper.

Genetic algorithms are an optimisation method that emulates the evolutionary principle of the survival of the fittest. This is because several possible solutions for our minimisation problem are handled simultaneously, as though each were an individual of a population; each iteration attempts to discard the poorest solutions and to improve the best already found. See for instance Jang (1997) for more details; the description of the particular genetic algorithm used, given below in section 7.1, also helps to make out how the optimisation is performed.

Genetic algorithms were chosen for they ensure reasonable results for nearly all minimisation problems. Other methods may perform better in particular cases, but knowing in advance which ones might is guesswork. On the other hand, genetic algorithms have fairly good results in almost all cases (Silva *et al.*, 2005).

7 Example

Let

$$G(s) = \frac{1}{39.69s^{1.26} + 0.598}. \tag{42}$$

This transfer function describes a thermal system heated by an electrical radiator (the input being a voltage) with the temperature measured by a pyrometer (the output being a voltage too) (Vinagre *et al.*, 2001).

The parameters of (42) have been identified by numerically fitting its step response to experimental values. So there are no reasons why the much more tractable commensurate (Q = 4) transfer function

$$G(s) = \frac{1}{39.69s^{5/4} + 0.598} \tag{43}$$

should not be used instead, its step and frequency responses being indistinguishable from those of (42). Suppose that we model (white, 0.01 V^2 intensity) noise as affecting both input (**L** = **B** in Fig. 1) and output. We want the output to remain unchanged in spite of noise, with the transfer function from w_1 to z_1 smaller than -6 dB over all frequencies and the (not weighted) transfer function from w_2 to z_1 decaying significantly (say, at -40 dB/decade at least) for high frequencies (say, above 1 rad/s, given the nature of the plant). The transfer functions mentioned are those without weights, SG_1 and SG_2K.

After some trial and error, the following weights have been selected:

$$\begin{aligned} W_1(s) &= \frac{1}{0.01}\frac{0.1s^{1/4} + 10}{s^{1/4} + 1.25} \\ W_2(s) &= \frac{1}{0.01}\frac{1429s^{1/4} + 5000}{s^{1/4} + 1000}. \\ W_3 &= W_4 = 1 \end{aligned} \tag{44}$$

These weights are fractional-order transfer functions, but integer-order weights might have been used instead. For this particular problem, fractional-order weights allowed attaining the control objectives more easily, but this is not always necessarily so. Integer-order weights have the additional advantage of having frequency responses easier to obtain. Furthermore, even though in this particular case a single set of weights sufficed for both the H_2 and the H_∞ controllers, this is not always necessarily so: different weights might have been necessary.

7.1 Algorithm

The algorithm to find a controller was as follows:

▪ A population with fifty individuals is created. Each individual is a transfer function matrix with a dimension compatible with the dimensions of the plant (in this case, controllers are SISO). The orders of the numerators and the denominators are those of the plant or the ones immediately above or below. Parameters are stored as real numbers.

▪ The H_2 or H_∞ norm of the matrix transfer function in (38) is evaluated for all individuals (in this case, this is a 2×2 matrix). The smaller the norm, the fitter the individual is.

▪ A new population is created with 90% of the size of the original one. Individuals are selected for this group according to their fitness. These will be the parents in the next step.

▪ The parents are recombined and replaced by their offspring. In other words, parents are matched in pairs; each pair is replaced by two new individuals, called the offspring; the parameters of each of the offspring are randomly chosen from those of its parents.

▪ The offspring undergo a mutation. In other words, some of their parameters, randomly chosen, are changed by addition of random values. The mutation probability is such that the average number of mutated parameters per individual is 0.5.

▪ These mutated descendents replace the less fit individuals in the original population. The resulting population is used for a new iteration, beginning with the evaluation of the norms, as explained in the second step above.

▪ Iterations stop after a certain maximum number of iterations (500 in this case) or after a certain maximum number of iterations without improvement in the results (in this case 50 for the H_2 norm and 30 for the H_∞ norm; this last value was smaller because calculations were slower).

The Genetic Algorithm Toolbox for Matlab (Chipperfield *et al.*, 1994) was used to implement this algorithm.

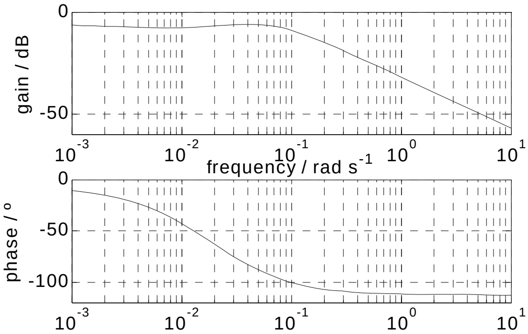


Figure 2
Bode diagram of SG_1 for H_2 controller (45)

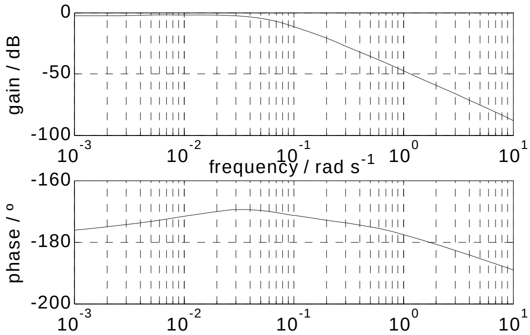


Figure 3
Bode diagram of SG_2K for H_2 controller (45)

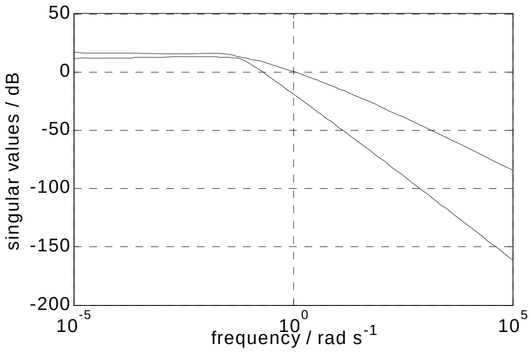


Figure 4
Singular values of loop transfer function (38) for H_2 controller (45)

7.2 Results

The following H_2 controller was found, with a norm of (38) equal to 1.7905:

$$K(s)=\frac{5.704}{s^{5/4}+10s+10s^{3/4}+9.999^{1/2}+9.422s^{1/4}-5.399} \quad (45)$$

The relevant Bode and singular value plots are found in Figure 2, Figure 3 and Figure 4.

The following H_∞ controller was found, resulting in a norm of (38) equal to 6.7115:

$$K(s)=\frac{7.448}{s^{5/4}+9.383s+8.642s^{3/4}-2.316s^{1/2}+9.227s^{1/4}-4.736} \quad (46)$$

The relevant Bode and singular value plots are found in Figure 5, Figure 6 and Figure 7.

7.3 Discussion

For both controllers, the magnitude of SG_1 is always below -6 dB (actually its maximum is -6.05 dB for the H_2 controller and -7.13 dB for the H_∞ controller). SG_2K decays with -45 dB/decade with both controllers. This means the objectives were attained in both cases. This is also shown by simulations of the resulting control loops. Changes in plant parameters are also handled by the resulting controllers.

Other values for weights W_1 to W_4 allow obtaining different results and shaping the loop in other ways. Thus it would be possible to cope with different performance specifications.

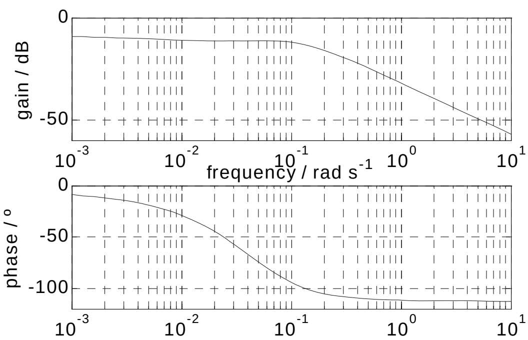


Figure 5
Bode diagram of SG_1 for H_∞ controller (46)

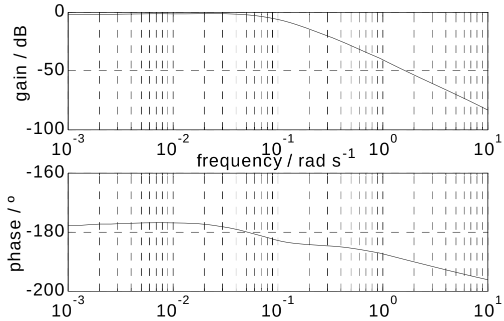


Figure 6

Bode diagram of SG_2K for H_∞ controller (46)

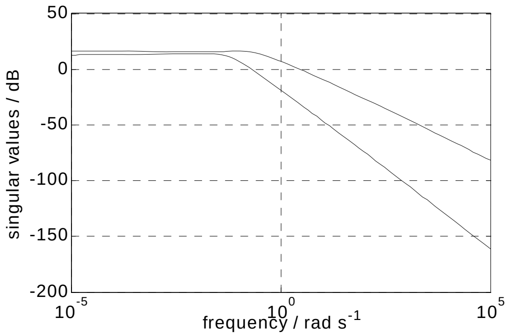


Figure 7

Singular values of loop transfer function (38) for H_∞ controller (46)

Two important questions arise from the use of a numerical minimisation method. The first is the time needed to reach a solution. For the H_2 case of the problem above, this means about 1 minute and 38 seconds of computation per iteration are needed (in a Pentium IV @ 2.53 GHz); for the H_∞ case, this may mean up to about 2 minutes and 50 seconds per iteration (in the same machine), but depends on how close is the mesh of frequencies used to estimate the norm. These are bearable values, though faster results would, of course, be desirable.

Still concerning this point, it is worth noticing that an increase in the dimension of (38) reflects very heavily on the computational effort needed. Dimensions above 4 often prevent the numerical algorithm from reaching a solution.

The second question concerns how far the optimisation went when it stops. The best validation possible is to use the algorithm for integer plants, for which there is an analytical solution available, and then compare both results. This was tried for several cases, and the numerical method always got very close to the analytical result. Assuming the same to happen with the fractional case, it seems that further possible reductions in norms are not very relevant.

Conclusion

In this paper the task of finding H_2 and H_∞ controllers for fractional plants was successfully addressed. This involves finding the norms of transfer function matrixes. A numerical optimisation method (a genetic algorithm) was used, given the absence of known analytical methods.

The drawbacks of this way of dealing with the problem are long simulation times (with the magnitude of hours) and the impossibility of knowing for sure how far the optimisation went. Thus, and beyond checking the validity of these results, the future work in this area consists in looking for analytic methods of developing these controllers.

References

[1] A. Chipperfield, P. Fleming, H. Pohlheim, C. Fonseca: Genetic Algorithm Toolbox for Use with Matlab, 1994, Available: http://www.shef.ac.uk/~gaipp/ ga-toolbox/

[2] J. Doyle, K. Glover, P. Khargonekar, B. Francis: State-Space Solutions to Standard H_2 and H_∞ Control Problems, *IEEE Transactions on Automatic Control*, Vol. 34, No. 8, pp. 831-847, Aug. 1989

[3] I. Gradshteyn, I. Ryzhik: *Table of Integrals, Series and Products*. Boston, Academic Press, 1980

[4] J.-S. R. Jang: Derivative-free optimization, in *Neuro-fuzzy and Soft Computing*. Upper Saddle River: Prentice Hall, 1997, pp. 173-196

[5] L. Lublin, S. Grocott, M. Athans: H_2 (LQG) and H_∞ control, in *The control handbook*. Boca Raton: CRC Press, 1996, pp. 635-650

[6] R. Malti, M. Aoun, O. Cois, A. Oustaloup, F. Levron: H_2 Norm of Fractional Differential Systems, in *ASME Symposium on Fractional Derivatives and Their Applications*, Chicago, 2003

[7] K. Miller, B. Ross: *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: John Wiley and Sons, 1993

[8] I. Petras, M. Hypiusova: Design of Fractional-Order Controllers via H_∞ Norm Minimisation, in *Selected Topics in Modelling and Control*, Slovak University of Technology Press, Bratislava, 2002, pp. 50-54

[9] I. Podlubny: *Fractional Differential Equations*. San Diego: Academic Press, 1999

[10] S. Samko, A. Kilbas, O. Marichev: *Fractional Integrals and Derivatives: Theory and Applications*. Yverdon: Gordon and Breach Science Publishers, 1993

[11] C. A. Silva, J. M. Sousa, T. Runkler, J. Sá da Costa: A Logistic Process Scheduling Problem: Genetic Algorithms or Ant Colony Optimization?, in *IFAC 2005*, to be published

[12] B. M. Vinagre, I. Petras, P. Merchan, L. Dorcak: Two Digital Realizations of Fractional Controllers: Application to Temperature Control of a Solid, in *European Control Conference*, Porto, 2001, pp. 1764-1767