Precautionary Saving with Possibilistic Background Risk

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Abstract—Possibility risk theory starts from the hypothesis that this is described by a fuzzy number and not by a random variable, as in the traditional probabilistic modeling. This paper is an attempt of possibilistic approach to risk management problem. In particular, the models proposed in this paper can be connected by the way the possibilistic risk influences the probability of a loss, the dimensions of the loss or both of them simultaneously considered (i.e. self-insurance, self-protection, or self-insurance-cum-protection). In the paper there are studied two-period models of risk management in which the possibilistic risk may appear in the first period or in the second period. The main results of the paper establish necessary and sufficient conditions, or only sufficient, for the optimal management level effort to raise or to decrease as a result of adding such a possibilistic risk.

I. INTRODUCTION

Prevention consists in the measures taken by a decision-maker (agent) in order to minimize the effects of a loss which may occur in an economic and financial activity with risk parameters. According to [15], [6], risk is identified with the size and the probability of the loss, and precaution is the effort to reduce each of them in part or simultaneously considered. In the known terminology, self-protection is the reduction of the probability of loss, self-insurance is the reduction of the size of loss and self-insurance-cum-protection is the reduction of both of them.

The first study on prevention is the paper [15] by Ehrlich and Becker, in which the relationship between market insurance, self-insurance and self-protection. After that, several authors have studied the optimal prevention in one period models, or two period models (see [6] for a general discussion of these models). Some models reflected the relationship between risk aversion and prevention [8], [24], and others the prudence impact on optimal prevention [13], [9].

An important direction of the field deals with a background risk impact on prevention. For instance, in [1] the influence of a background risk on the relationship between risk-aversion and self-insurance is analyzed. In papers [5], [8], in the context of some two-period models there is emphasized the way various types of background risk modify the optimal self-protection level. The topic is tackled in a more general context in [21]: a risk management model including self-insurance and self-protection is proposed, then it is studied how the background risk produces changes on the management effort.

In all these approaches, risk is probabilistically treated, being represented by random variables. In the possibility theory of risk, this is modeled by fuzzy numbers (see [17]). Paper [19] is an attempt to treat prevention in a possibilistic context. In the extension of the paper [5] by Courbage and Ray, the models in [19] illustrate how possibilistic and mixed background risk operate on the optimal effort in the self protection activity.

This paper is an attempt to approach the issue from [21] by possibility theory. Our models start also from the benchmark two-period model of [21]. They correspond to the situations in which in one period or in both periods risks represented by random numbers are added. The optimization problems associated with these models are formulated by total utility functions whose expression incorporates forms of the possibilistic expected utility, a notion frequently used in [17].

Let us describe shortly the content of this paper. In Section 2 the definition of possibilistic expected utility and some of its basic properties are recalled from [17]. The construction of the management models from the paper will be based exactly on this notion of possibilistic expected utility. Section 3 presents the benchmark two-period model from [21]: the entities from which its construction starts, the total utility function form and the first order conditions corresponding to the optimization problem.

Sections 4 and 5 contain the two risk management models of the paper: the first model describes the situation in which the possibilistic risk appears in period 1, and the second model describes the case in which the possibilistic risk is present in the second period. Optimization problems of the models are formulated, the corresponding first-order conditions and there are proved various results which establish necessary and sufficient conditions or only sufficient on the decrease or increase of the management effort level.

The presence of possibilistic risk in one of the two periods or in both periods makes the optimal management effort level to raise or to decrease with respect to the management effort level from the benchmark model. The main results of the paper establish necessary and suffi-
cient conditions or only sufficient, for the optimal management effort level to raise or to decrease.

The model of Section 6 treats the case when the possibilistic risk is added both in period 1 and in period 2.

II. POSSIBILISTIC EXPECTED UTILITY: A BRIEF DESCRIPTION

All the models from this paper will be developed in a framework defined by the following components:

- a continuous utility function $u : \mathbb{R} \to \mathbb{R}$, representing an agent;
- a fuzzy number $A$ modelling a possibilistic risk;
- a weighting function $f : [0, 1] \to \mathbb{R}$ ($f$ is non-negative, monotone increasing and verifies the normality condition $\int f(\gamma) d\gamma = 1$).

For the rest of the paper we fix a weighting function $f$. We will denote by $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$ the level sets of the fuzzy number $A$.

To the context defined by the three components above we associate the following notion of possibilistic expected utility [17]:

$$E_f(u(A)) = \frac{1}{2} \int_0^1 [u(a_1(\gamma)) + u(a_2(\gamma))] f(\gamma) d\gamma$$

(2.1)

Particularizing in (2.1) $u = 1_{\mathbb{R}}$ (the identity of $\mathbb{R}$) we obtain the known expression of possibilistic expected utility [2], [3].

$$E_f(A) = \frac{1}{2} \int_0^1 [a_1(\gamma) + a_2(\gamma)] f(\gamma) d\gamma$$

(2.2)

The notion of possibilistic expected utility (2.1) and the possibilistic indicators derived from it (possibilistic expected value, possibilistic variance, possibilistic covariance, etc.) allow for a possibilistic treatment of some topics or risk theory (risk aversion, prudence, saving, etc.) (see [4], [17], [18]).

We recall some important properties of possibilistic expected utility.

Proposition 2.1: [17] Let $f : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be two continuous utility functions and $a, b \in \mathbb{R}$.

(i) If $u = ag + bh$ then $E_f(u(A)) = aE_f(u(g)) + bE_f(u(h))$;

(ii) If $g \preceq h$ then $E_f(g(A)) \leq E_f(h(A))$.

The linearity property (i) and the monotonicity property (ii) will be applied in the computations of the following sections without an explicit mention.

Proposition 2.2: [17] If the utility function $u$ is convex, then $u(E_f(A)) \leq E_f(u(A))$.

The Jensen type inequality from the previous proposition will be used to establish some necessary and sufficient conditions to indicate when the management activity level increases or decreases as a result of the possibilistic background risk presence.

III. THE BENCHMARK MODEL

This section contains a general two-periods model of risk management with separable utility (cf. [17], Section 2). In this model, management activities such as self-protection or self-insurance, separately or simultaneously analyzed, may be interpreted.

One considers an agent whose management activities are carried out in two period 1 and 2: in period 1 a management effort whose effect is manifested in period 2 by changes in the probability of the occurrence of a loss and changes in the measure of this loss.

This management activity is described in [17] by a model defined by the following entities:

- $w_1$ and $w_2$ are the wealth in period 1 and 2;
- $u$ and $v$ are utility functions for periods 1 and 2;
- $e$ is the level of management effort (one assumes $e > 0$);
- $c(e)$ is the cost associated with the effort $e$;
- $I(e)$ is the size of loss corresponding to $e$;
- $p(e)$ is the probability of loss at an effort level $e$.

On the functions defining the model the following assumptions are made:

- the utility functions $u$ and $v$ have the class $C^2$, $u' > 0$, $v' > 0$, $u'' < 0$, $v'' < 0$;
- $c(0) = 0$, $c' > 0$, $c'' > 0$;
- $p' \leq 0$, $p'' > 0$;
- $l' \leq 0$, $l'' \geq 0$.

The overall utility function of the model is

$$V(e) = u(w_1 - c(e)) + p(e)v(w_2 - l(e)) + (1 - p(e))v(w_2)$$

(3.1)

Following [17], the notations are introduced:

$$u_1(e) = u(w_1 - c(e)); w_{2L} = v(w_2 - l(e)); v_{2N} = v(w_2)$$

(3.2)

By deriving (3.1) one obtains:

$$V'(e) = -c'(e)u_1'(e) + p'(e)v_{2L}(e) - v_{2N} - p(3)p''(e)v_{2L}(e)$$

(3.3)

A simple computation shows that $V''(e) < 0$ for any $e$ (see [17]). We consider the optimization problem:

$$\max V(e) = V(e^*)$$

(3.4)

in which $e^*$ is the management effort level for which the maximum of the utility $V(e)$ is achieved. The solution $e^*$ of (3.4) will fulfill the first-order condition [17]:

$$-c'(e^*)u_1'(e^*) + p'(e^*)[v_{2L}(e^*) - v_{2N}] - p(3)p''(e^*)v_{2L}(e^*) = 0$$

(3.5)

All the models from the next sections will be obtained from the benchmark model by adding a possibilistic background risk.

IV. MODEL 1: WITH POSSIBILISTIC BACKGROUND RISK IN PERIOD 1

This model takes into account the situation in which in period 1 there is a possibilistic background risk represented by a fuzzy number $A$, and in period 2 no background risk appears.

The total utility function will be:

$$V_1(e) = E_f(u(w_1 - c(e) + A)) + p(e)v(w_2 - l(e)) + (1 - p(e))v(w_2)$$

(4.1)

We will assume that the level sets of the fuzzy set $A$ are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for any $\gamma \in [0, 1]$. By the definition (2.1) of the possibilistic expected utility, $V_1(e)$ is written as:

$$V_1(e) = \frac{1}{2} \int_0^1 [u(w_1 - c(e) + a_1(\gamma)) + u(w_1 - c(e) + a - 1(\gamma))] f(\gamma) d\gamma + p(e)v(w_2 - l(e)) + (1 - p(e))v(w_2)$$

(4.2)

By derivation, from (4.2) one obtains
**V**'_2(ε) = \frac{-c_l^1}{\sqrt{2}} \int_0^1 [u'(w_1 - c_l(ε) + a_l(γ)) + u'(w_1 - c_l(ε) + a_l(γ))] f(γ) dγ + p(ε)[v(w_2 - l(ε)) - p(ε')v'(w_2 - l(ε))] f(γ) dt.

V'_1(ε) = -c_l^1(ε)E_f(\{u'_1^1(ε) + A\}) + p(ε')[v_{2L}(ε) - v_{2N} - p(ε')l'(ε)v_{2L}(ε)].

Doing as above, by deriving (4.3) one obtains:

V'_2(ε) = -c_l^2(ε)E_f(\{u'_1^2(ε) + A\}) + p(ε')[v_{2L}(ε) - v_{2N} - p(ε')l'(ε)v_{2L}(ε)].

Since V is strictly increasing, v_{2L}(ε) - v_{2N} = v(w_2 - l(ε) - v(w_2) < 0. Then it follows immediately that V'_1(ε) < 0 for any ε. We consider the optimization problem:

\max V'_1(ε) = V'_1(ε^*)

in which ε^* is the effort level maximizing the utility V'(ε).

According to (4.3), the first-order condition associated to the optimization problem (4.4) takes the form:

\[ c_l^1(ε^*)E_f(\{u'_1^1(ε^*) + A\}) = p(ε^*)[v_{2L}(ε^*) - v_{2N} - p(ε^*)l'(ε)v_{2L}(ε^*)] \]

**Proposition 4.1:** With the above notations, the following assertions are equivalent:

(a) \[ ε^* ≤ ε^* \]

(b) \[ u'_1^1(ε^*) ≤ E_f(u'_1^1(ε^*) + A). \]

**Proof:** Taking into account (4.3), (3.5) and the fact that V'_1 is strictly decreasing (since V'_1(ε) < 0), the equivalence of the following properties holds:

- \[ ε^* ≤ ε^* \]
- \[ V'_1(ε^*) ≤ V'_1(ε^* st t) = 0; \]
- \[ -c_l^1(ε^*)E_f(u'_1^1(ε^*) + A) + p(ε^*)[v_{2L}(ε^*) - v_{2N} - p(ε^*)l'(ε)v_{2L}(ε^*)] ≤ 0; \]
- \[ -c_l^1(ε^*)E_f(u'_1^1(ε^*) + A) + p(ε^*)[v_{2L}(ε^*) - v_{2N} - p(ε^*)l'(ε)v_{2L}(ε^*)] ≤ 0; \]
- \[ u'_1^1(ε^*) ≤ E_f(u'_1^1(ε^*) + A). \]

For the equivalence of the last two assertions we took into account that c_l^1(ε^*) > 0.

**Corollary 4.2:** If u'' > 0 and E_f(A) = 0 then ε^* ≤ ε^*.

**Proof:** Taking into consideration that u'_1 is convex we can apply Proposition 2.2.

\[ u'_1^1(ε^*) = u'_1^1(ε^* + E_f(A)) = u'_1^1(E_f(ε^* + A)) \]

thus ε^* ≤ ε^*, by Proposition 4.1.

Proposition 4.1 gives us a necessary and sufficient condition such that, by adding a possibilistic risk A in period 1, the level of optimal management effort decreases. Corollary 4.2 shows that if the agent is prudent in period 1, then the optimal effort level decreases in the presence of a possibilistic background risk in period 1.

**V. MODEL 2: WITH POSSIBILISTIC BACKGROUND RISK IN PERIOD 2**

The model of this section describes a risk management activity without background risk in period 1 and with background risk in period 2, represented by a fuzzy number B.

The total utility function of the model will be:

V_2(ε) = w(u(w_1 - c(ε)) + p(ε)E_f(v(w_2 - l(ε) + B)) + (1 - p(ε))E_f(v(w_2 + B))).

According to (2.1), V_2(ε) is written as:

V_2(ε) = u(w_1 - c(ε)) + \frac{p(ε)}{2} \int_0^1 [v(w_2 - l(ε) + b_1(γ)) + v(w_2 + b_2(γ))] f(γ) dγ.

By deriving (5.2) and applying again (2.1) one obtains:

V_2(ε) = -c_l^2(ε)E_f(\{u'_1^2(ε) + A\}) + p(ε')[v_{2L}(ε) - v_{2N} - p(ε')l'(ε)v_{2L}(ε)] - p(ε')E_f(v'(w_2 - l(ε) + B)) - p(ε')E_f(v(w_2 + B)).

which, with the notations (3.2), can be written:

V_2(ε) = -c_l^2(ε)E_f(\{u'_1^2(ε) + p(ε')E_f(v_{2L}(ε) + B) - E_f(v_{2N}(B)) - p(ε')l'(ε)E_f(v_{2L}(ε) + B) \}

By a standard calculation, one proves that V_2(ε) > 0 for any ε. We consider the optimization problem of the model:

\max V_2(ε) = V_2(ε^*)

where ε^* is the effort level maximizing V_2(ε). By (5.3), the first-order condition associated with the model will be:

\[ c_l^2(ε^*)u'_1^2(ε^*) = p(ε^*)[E_f(v_{2L}(ε^*) + B) - E_f(v_{2N}(B))] - p(ε^*)l'(ε)E_f(v_{2L}(ε^*) + B). \]

**Proposition 5.1:** The following assertions are equivalent:

(a) \[ ε^* > ε^* \]

(b) \[ p(ε^*)[E_f(v_{2L}(ε^*) + B) - E_f(v_{2N}(B))] - p(ε^*)l'(ε)E_f(v_{2L}(ε^*) + B) ≤ 0. \]

**Proof:** One notices that V_2 is strictly decreasing (since V_2'' < 0). Then, by taking into account (5.3) and (3.5), the following properties are equivalent:

- \[ ε^* ≥ ε^* \]
- \[ V_2(ε^*) ≥ V_2(ε^*) \]
- \[ -c_l^2(ε^*)u'_1^2(ε^*)[E_f(v_{2L}(ε^*) + B) - E_f(v_{2N}(B))] - p(ε^*)l'(ε)E_f(v_{2L}(ε^*) + B) ≥ 0; \]
- \[ -p(ε^*)[v_{2L}(ε^*) - v_{2N}] - p(ε^*)l'(ε)v_{2L}(ε^*) + p(ε^*)E_f(v_{2L}(ε^*) + B) - E_f(v_{2N}(B)) - p(ε^*)l'(ε)E_f(v_{2L}(ε^*) + B) ≥ 0. \]

Proposition 5.1 is a necessary and sufficient condition for the presence of possibilistic risk B of the management activity in period 2 to lead to the raise of the effort level. We will prove next that the inequality (b) is satisfied under the conditions of the agent’s prudence in period 2 (v'' > 0).

**Lemma 5.2:** If v'' > 0 and E_f(A) = 0 then the following inequalities are true:

(i) \[ E_f(v_{2L}(ε^* + B)) - v_{2L}(ε^*) ≥ 0; \]
(ii) \[ E_f(v_{2L}(ε^*) + B) - E_f(v_{2N}(B)) - v_{2L}(ε^*) - v_{2N} ≤ 0. \]

**Proof:** (i) By applying Proposition 2.2 and taking into account that E_f(B) = 0 it follows:

\[ V_{2L}(ε^*) = v'(w_2 - l(ε^*)) = v'(E_f(w_2 - l(ε^*) + B)) ≤ E_f(v'(w_2 - l(ε^*) + B) = E_f(v_{2L}(ε^*) + B). \]

(ii) From v'' > 0 it follows that v'''' > 0 and the inequality (ii) follows immediately.

**Corollary 5.3:** If E_f(B) = 0 and v'''' > 0 then ε'''' ≥ ε^*.
VI. MODEL 3: WITH POSSIBILISTIC BACKGROUND RISK IN BOTH PERIODS

The model of this section describes a management activity in which the possibilistic background risk is present in both periods: in period 1 it is represented by a fuzzy number $A$, in the second period by a fuzzy number $B$.

Assume that the level sets of the fuzzy numbers $A, B$ have the form $[A]_a = [a_1(\gamma), a_2(\gamma)]$ and $[B]_a = [b_1(\gamma), b_2(\gamma)]$ for any $\gamma \in [0,1]$. The total utility function of the model will be:

$$V_3(e) = E_f(u(w_1 - c(e) + A)) + p(e)E_f(v(w_2 - l(e) + B)) + (1 - p(e))E_f(v(w_2 + B)). \quad (6.1)$$

According to (2.1), $V_3(e)$ can be written as:

$$V_3(e) = \frac{1}{2} \int_0^1 [u(w_2 - c(e) + a_1(\gamma))] + u(w_2 - c(e) + a_2(\gamma))]f(\gamma)d\gamma \frac{1}{2} \int_0^1 [v(w_2 - l(e) + b_1(\gamma))] + v(w_2 - l(e) + b_2(\gamma))]f(\gamma)d\gamma.$$

Deriving the previous formula and taking into account (2.1) one obtains:

$$V_3''(e) = -\frac{d'(e)}{d\gamma}E_f(u'(w_1 - c(e) + A)) + p'(e)E_f(v(w_2 - l(e) + B)) - p(e)E_f(v'(w_2 - l(e) + B)) - p(e)E_f(v'(w_2 - l(e) + B)) + p'(e)E_f(v(w_2 - l(e) + B)) + p'(e)E_f(v(w_2 - l(e) + B)).$$

By deriving one more time the above relation it follows $V_3'''(e)$.

The main result establishes a necessary and sufficient condition such that, as a result of adding a possibilistic risk in both periods 1 and 2, the maximal effort level raises.

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