LMI Approach to Non-Lyapunov Stability of Discrete Descriptor Time Delay Systems

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Abstract—The article provides sufficient conditions for the practical and finite time stability of linear singular continuous time delay systems fulfilling the following formulation: $E\dot{x}(k+1)=A_0x(k)+A_1x(k-1)$. Analyzing the finite time stability concept, novel delay independent conditions have been presented. The conditions were derived using the linear matrix inequality (LMI) approach. The LMI method have been presented. The conditions were derived using the formulation:

$$x(I) + F_0 x(I) + F_1 x(I) = 0$$

for the finite time stability based on the LMI approach has been established.

I. INTRODUCTION

Both dynamic and static characteristics of certain control systems in the state-space formulation needs to be analyzed at the same time. The systems described by differential and algebraic equations are referred to as singular systems. Singular systems are also known as degenerate, descriptor, generalized, differential-algebraic or semi-state systems. Recently, singular systems have become one of the important research topics in control theory. During the past three decades, singular systems have attracted much attention due to their comprehensive applications in various fields, such as economics (the Leontief dynamic model), modeling in electrical engineering [1], and mechanical modeling, [2].

Stability on finite and infinite time intervals of the singular systems deserves specific attention. Stability analyses have been tightly connected with the investigation of the solution properties such as uniqueness and existence. Consistent initial conditions and solutions in the state-space and phase-space have been widely investigated especially when the solutions were derived based on discrete fundamental matrices [3]. For singular systems the consistent initial conditions denoted by $x_0$, can generate a solution sequence $(x(k); k\geq 0)$ which usually has a practical meaning in the models of physical systems. Some of the previous questions do not need to be considered for the regular (normal) systems.

The problem of investigation of time delay systems has been exploited over many years. Time delay is often encountered in various technical systems, such as in electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause an undesirable system transient response or even instability. It should be emphasized that there are a lot of systems that have the phenomena of time delay and singularity simultaneously. Such systems are called the singular systems with time delay or descriptor time delay systems in a discrete case. The stability, as a special characteristic of these systems, will be analyzed in the following part. However, practical matters require us to concentrate not only on the system stability (e.g. in the sense of Lyapunov), but also on the bounds of system trajectories.

A system could be stable but still completely useless, because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined a priori in a given problem. Besides that, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval. These boundedness properties of system responses, i.e. the solution of system models, are important from the engineering point of view.

To the best of the authors’ knowledge, so far there have been no results concerning the non-Lyapunov stability of the discrete descriptor time delay systems.

II. BASIC NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\Re$</td>
<td>Real vector space</td>
</tr>
<tr>
<td>$C$</td>
<td>Complex vector space</td>
</tr>
<tr>
<td>$I$</td>
<td>Unit matrix</td>
</tr>
<tr>
<td>$F= (f_{ij}) \in \Re^{m \times n}$</td>
<td>Real matrix</td>
</tr>
<tr>
<td>$F^T$</td>
<td>Transpose of matrix $F$</td>
</tr>
<tr>
<td>$F &gt; 0$</td>
<td>Positive definite matrix</td>
</tr>
<tr>
<td>$F \geq 0$</td>
<td>Positive semi definite matrix</td>
</tr>
<tr>
<td>$\Re(F)$</td>
<td>Range of matrix $F$</td>
</tr>
<tr>
<td>$\Re(F)$</td>
<td>Null space (kernel) of matrix $F$</td>
</tr>
<tr>
<td>$\lambda(F)$</td>
<td>Eigenvalue of matrix $F$</td>
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</table>
\[ \sigma(\lambda) (F) \] Singular value of matrix \( F \)

\[ \rho(F) \] Spectral radius of matrix \( F \)

\[ \| F \| \] Euclidean matrix norm

\[ \| F \| = \sqrt{\lambda_{\text{max}} (A^T A)} \]

\( F^D \) Drazin inverse of matrix \( F \)

\( \Rightarrow \) Follows

\( \mapsto \) Such that

### III. NON-LYAPUNOV STABILITY OF LINEAR DISCRETE DESCRIPTOR TIME DELAY SYSTEM – PREVIOUS RESULTS

Consider a linear discrete descriptor time delay system (LDDTDS), with state delay, described by:

\[ E x(k+1) = A_0 x(k) + A_1 x(k-h) , \quad x(k_0) = \psi (k_0) , \quad -1 \leq k_0 \leq 0 \]

where \( x(k) \in \mathbb{R}^n \) is a state vector. The matrix \( E \in \mathbb{R}^{m \times n} \) is a necessarily singular matrix, with property \( \text{rank} \, E = r < n \) and with matrices \( A_0 \) and \( A_1 \) of appropriate dimensions.

For LDDTDS, (1), the following definitions are valid [4].

**Definition 1:** The LDDTDS is said to be regular if 
\[ \det \{ z^2 E - z A_0 - A_1 \} \neq 0 \]

**Definition 2:** The LDDTDS is said to be causal if it is regular and:

\[ \deg \{ z^n \det \{ zE - A_0 - z^{-1} A_1 \} \} = n + \text{rank} \, E. \]

**Definition 3:** The LDDTDS is said to be stable if it is regular and

\[ \rho \left( E, A_0, A_1 \right) \subset D(0,1) \]

\[ \rho \left( E, A_0, A_1 \right) = \{ z : \det \{ z^2 E - z A_0 - A_1 \} = 0 \} \]

**Definition 4:** The LDDTDS is said to be admissible if it is regular, causal and stable.

Let us consider a linear discrete system with state delay described by:

\[ E x(k+1) = A_0 x(k) + \sum_{j=1}^{M} A_j x(k-h_j) \]

\[ x(\vartheta) = \psi (\vartheta) \quad \vartheta \in \{-N,-(N+1),\ldots,0\} \]

where \( x(k) \in \mathbb{R}^n \), is state vector \( E, A_j \in \mathbb{R}^{m \times n}, j = 1, \ldots, M ; h_j, j = 1, \ldots, M \) are integers representing the system time delay, \( N = \max \{ h_1, h_2, \ldots, h_M \} \) and \( \psi (\cdot) \) is a priori known vector function of initial conditions.

Let \( \mathbb{R}^n \) denote the state space of the systems given by (1–2) and \( \| \cdot \| \) Euclidean norm.

**Definition 5:** LDDTDS is said to be causal system, given by (2), is finite time stable with respect to \( \{ k_0, K_N, S_x, S_{\alpha}, R \} \), with \( R > 0 \), and if only if \( \forall x_0 \in W^*_{\text{div}} \) satisfying:

\[ \| x(k) \|_{E^T E}^2 < \alpha \]

\[ \implies \| x(k) \|_{E^T E}^2 < \beta, \quad \forall k \in K_N \]

\( W^*_{\text{div}} \) is the subspace of consistent initial conditions.

**Definition 6:** The causal system given by (1), is practically unstable with respect to \( \{ k_0, K_N, \alpha, \beta, R \} \), \( \alpha < \beta \), if and only if \( \exists \psi \in W^*_{\text{div}} \) such that

\[ \| x_0 \|_{E^T E}^2 < \alpha \]

\[ \implies \| x(k) \|_{E^T E}^2 < \beta \]

for some \( k^* \in K_N \).

**Definition 7:** The causal system given by (1), is attractive practically stable with respect to \( \{ k_0, K_N, S_x, S_{\alpha}, R \} \), if and only if \( \forall x_0 \in W^*_{\text{div}} \) satisfying:

\[ \| x(k_0) \|_{G=E^T E}^2 \leq \| x_0 \|_{G=E^T E}^2 < \alpha \]

\[ \implies \| x(k) \|_{G=E^T E}^2 < \beta, \quad \forall k \in K_N \]

with property that:

\[ \lim_{k \to \infty} \| x(k) \|_{G=E^T E}^2 \to 0 \]

\( W^*_{\text{div}} \) is the subspace of consistent initial conditions.
Remark 1: The singularity of matrix $E$ will ensure that solutions to (1) exist for only special choices of $x_0$.

In [5] the subspace of $W_{\text{dis}}^*$ of consistent initial conditions is shown to be the limit of the nested subspace algorithm:

$$W_{\text{dis}}^* = R^n,$$

where

$$W_{\text{dis}}^* = A_0^{-1} \left( EW_{\text{dis}}^* (k) \right)_{A_1=0}, \quad k \geq 0.$$  \hspace{1cm} (3)

Moreover, if $x_0 \in W_{\text{dis}}^*$ then $x(k) \in W_{\text{dis}}^*$, $\forall k \geq 0$ and $(\lambda E - A_0)_{A_1=0}$ is invertible for some $\lambda \in \mathbb{C}$ (a condition for uniqueness), then:

$$W_{\text{dis}}^* \cap \mathbb{R}(E) = \{0 \}.$$  \hspace{1cm} (4)

Remark 2: Note that here $G = E^T R E \geq 0$, where $R = R^T > 0$ is an arbitrarily matrix.

Also note that (4) implies that:

$$\| x(k) \|_{E^T R E} = \sqrt{x^T(k) E^T R E x(k)},$$

is a norm on $W_{\text{dis}}^*$.

Remark 3: For the future purpose we use the following definitions of the smallest, respectively the largest eigenvalues of matrix $R = R^T$, with respect to subspace of consistent initial conditions $W_{\text{dis}}^*$ and matrix $G$.

Proposition 1: If $x^T(t) R x(t)$ is a quadratic form on $R^n$, then it follows that there exist numbers $\lambda_{\min}(R)$ and $\lambda_{\max}(R)$ satisfying:

$$-\infty \leq \lambda_{\min}(R) \leq \lambda_{\max}(R) \leq +\infty,$$

such that:

$$\lambda_{\min}(\Xi) \leq x^T(k) R x(k) \leq \lambda_{\max}(R),$$

$$\forall x(k) \in W_{\text{dis}}^* \setminus \{0\}. \hspace{1cm} (6)$$

with matrix $R = R^T$ and corresponding eigenvalues:

$$\lambda_{\max}(R, G, W_{\text{dis}}^*) = \min \left\{ x^T(k) R x(k): x(k) \in W_{\text{dis}}^* \setminus \{0\} \right\}, \hspace{1cm} (7)$$

$$\lambda_{\min}(R, G, W_{\text{dis}}^*) = \max \left\{ x^T(k) R x(k): x(k) \in W_{\text{dis}}^* \setminus \{0\} \right\}. \hspace{1cm} (8)$$

Note that $\lambda_{\min} > 0$ if $R = R^T > 0$.

B. Stability Theorems

Theorem 1: Suppose matrix $\left\{ A_1^T A_1 - E^T E \right\} > 0$. The causal system given by (1) is finite time stable with respect to $\left\{ k_0, K_N, \alpha, \beta, \| \cdot \| \right\}$, $\alpha < \beta$, if there exists a positive real number $p$, $p > 1$, such that:

$$\| x(k-1) \|_{\delta^T A_1} < p^2 \| x(k) \|_{\delta^T A_1}, \quad \forall k \in K_N,$$

$$\forall x(k) \in S, \quad \forall x(k) \in W_{\text{dis}}^* \setminus \{0\}$$

and if the following condition is satisfied,

$$\overline{x}_k (\cdot) < \frac{\beta}{\alpha}, \quad \forall k \in K_N,$$

where:

$$\overline{x}_k (\cdot) = \overline{x}_k \left( x^T(k) A_0^T \left[ I - A_1 \left( A_1^T A_1 - E^T E \right)^{-1} A_1^T + p^2 A_1^T \right] A_1 x(k) \right),$$

$$\forall x(k) \in W_{\text{dis}}^* \setminus \{0\}. \hspace{1cm} (11)$$

as in [3].

Theorem 2: Suppose matrix $\left\{ A_1^T A_1 - E^T E \right\} > 0$. The causal system (1) is finite time unstable with respect to $\left\{ k_0, K_N, \alpha, \beta, \| \cdot \| \right\}$, $\alpha < \beta$, if there exists a positive real number $p$, $p > 1$, such that:

$$\| x(k-1) \|_{\delta^T A_1} < p^2 \| x(k) \|_{\delta^T A_1}, \quad \forall k \in K_N,$$

$$\forall x(k) \in S, \quad \forall x(k) \in W_{\text{dis}}^* \setminus \{0\}$$

and if for $\forall k_0 \in W_{\text{dis}}^*$ and $\| x(k) \|_{G = E^T E} < \alpha$ there exists:

$$\lambda_{\max} (\Xi) > \frac{\beta}{\delta}, \quad k^* \in K_N \hspace{1cm} (13)$$

where:

$$\lambda_{\max} (\Xi) = \lambda_{\max} \left( x^T(k) A_0^T \left[ I - A_1 \left( A_1^T A_1 - E^T E \right)^{-1} A_1^T + 2p^2 \delta \right] A_1 x(k) \right),$$

$$\forall x(k) \in W_{\text{dis}}^* \setminus \{0\}. \hspace{1cm} (14)$$

[3].

Theorem 3: Suppose matrix $\left\{ A_1^T P A_1 - E^T P E \right\} > 0$. The causal system given by (1), with $\det A_0 \neq 0$, is attractive
practically stable} with respect to \( \left\{ k_0, K, \alpha, \beta, \| \cdot \| \right\} \), \( \alpha < \beta \), if there exists a matrix \( P = P^T > 0 \), being the solution of:
\[
A_i^TPA_0 - E^TPE = -2(Q + S), \quad (15)
\]
with matrices \( Q = Q^T > 0 \) and \( S = S^T \), such that:
\[
X^T(k)(Q + S)X(k) > 0, \quad \forall X(k) \in W_{d,k^*} \setminus \{0\}, \quad (16)
\]
is a positive definite quadratic form on \( W_{d,k^*} \setminus \{0\} \), \( p \) real number, \( p > 1 \), such that:
\[
\left\| X(k-1) \right\|_{A_i^TPA_0}^2 < p^2 \left\| X(k) \right\|_{A_i^TPA_0}^2, \quad \forall k \in K_N, \quad (17)
\]
\[\forall X(k) \in S_{p}, \quad \forall X(k) \in W_{d,k^*} \setminus \{0\}\]
and if the following conditions are satisfied, [3]:
\[
\left\| A_1 \right\| < \sigma_{\min} \left( \frac{1}{Q^2} \right), \quad \sigma_{\max} \left( Q^2 E^T P \right), \quad (18)
\]
and
\[
\Sigma_{\max} (x) < \frac{\beta}{\alpha}, \quad \forall k \in K_N, \quad (19)
\]
where:
\[
\Sigma_{\max} (x) = \max \left\{ \frac{1}{2} x^T(k) A_i^T P^2 \Lambda P^2 A_i x(k) \right\}, \quad (20)
\]
\[\max \left\{ x(k) \in W_{d,k^*}, \quad x^T(k)E^TPEx(k) = 1 \right\}
\]
\[\Lambda = \left( I - A_i^T A_i P A_i - E^TPE \right)^{-1} A_i^T + p^2 I \]
\[\exists [3].
\]

Remark 4: (15-16) are in a modified form derived from [5].

Remark 5: The asymptotic stability of (1) is guaranteed by (15) and (18) based on the conditions presented in [5-6].

IV. MAIN RESULTS – LMI APPROACH

For the following considerations Definition 5 is crucial. Now it is possible to give the sufficient conditions, under which system (2) will be regular, causal and finite time stable, simultaneously, without assumptions of regularity.

Theorem 4: The system (2) is causal and finite time stable with respect to \( \left\{ \alpha, \beta, K, \| \cdot \| \right\} \) \( \alpha < \beta \), if letting
\[
E^TPE = E^T R^2 \Pi R^2 E, \quad \text{there exists a positive scalar} \ \varphi > 0 \quad \text{and two positive definite matrices} \ \Pi \quad \text{and} \ Q \quad \text{such that the following conditions hold:}
\]
\[
E^TPE \geq 0, \quad (21)
\]
\[
\begin{align*}
\Xi &= \begin{pmatrix}
A_i^T P A_0 + Q + E^T PE - \varphi E^T PE & A_i^T P A_h \\
A_i^T P A_h & -(Q - A_i^T P A_h)
\end{pmatrix} < 0
\end{align*}
\]
\[
(\varphi + 1)^\top \begin{pmatrix}
\lambda_{\max}(\Pi) + h_{\max}(Q) \\
\lambda_{\max}(\Pi)
\end{pmatrix} < \frac{\beta}{\alpha}, \quad \forall k \in K_N \quad (23)
\]

Proof: Let us consider the following Lyapunov-like aggregation function:
\[
V(x(k)) = x^T(k) E^T P E x(k) + \sum_{j=k-h}^{k-1} x^T(j) Q x(j) \quad (24)
\]
Then, the backward difference along the trajectories of system (2) is:
\[
\Delta V(x(k)) = V(x(k+1)) - V(x(k))
\]
\[= x^T(k) (A_i^T P A_h + Q - E^T PE) x(k)
\]
\[+ x^T(k) A_i^T P A_h x(k-h) + x^T(k-h) A_i^T P A_h x(k)
\]
\[\leq \zeta_i^T(k) \Gamma \zeta_i(k)
\]
where:
\[
\zeta_i^T(k) = \begin{bmatrix}
x^T(k) \\ x^T(k-h)
\end{bmatrix}
\]
\[
\Gamma = \begin{pmatrix}
A_i^T P A_h + Q + E^T PE & A_i^T P A_h \\
A_i^T P A_h & -(Q - A_i^T P A_h)
\end{pmatrix} \quad (26)
\]
From (22) and (25), one can have:
\[
\Delta V(x(k)) = \zeta_i^T(k) \zeta_i(k)
\]
\[\leq \zeta_i^T(k) \Xi \zeta_i(k)
\]
\[= \zeta_i^T(k) \Xi \zeta_i(k) - \zeta_i^T(k) \begin{pmatrix}
-\varphi E^T PE & 0 \\
0 & 0
\end{pmatrix} \zeta_i(k)
\]
\[\leq \zeta_i^T(k) \Xi \zeta_i(k) + \varphi \xi_i^T(k) E^T PE \xi_i(k)
\]
\[< \varphi \xi_i^T(k) E^T PE \xi_i(k)
\]
\[< \varphi \xi_i^T(k) E^T PE \xi_i(k) + \sum_{j=k-h}^{k-1} x^T(j) Q x(j)
\]
\[= \varphi V(x(k))
\]
\[= \zeta_i^T(k) \Xi \zeta_i(k) < 0.
\]
Moreover, it can be stated:
\[
\Delta V(x(k)) = V(x(k+1)) - V(x(k))
\]
\[< \delta V(x(k)) \quad (28)
\]
so that:
\[
V(x(k+1)) < (\varphi + 1) V(x(k)). \quad (29)
\]
Applying iteratively (29), we obtain:
\[
V(x(k)) < (\varphi + 1)^k V(x(k-1))
\]
\[< (\varphi + 1)^{k-1} V(x(k-2))
\]
\[< (\varphi + 1)^{k-2} V(x(k-3))
\]
[30]
On the other hand, we have:

\[
V(x(0)) = x^T(0)E^TPE(x(0)) + \sum_{j=1}^{n} x^T(j)Qx(j)
\]

\[
x^T(0)E^TPEx(0) + \sum_{j=1}^{n} x^T(j)Qx(j)
\]

\[
\leq \lambda_{\text{max}}(\Pi)x^T(0)E^TREx(0) + \lambda_{\text{max}}(Q) \sum_{j=1}^{n} x^T(j)x(j)
\]

and using the basic assumption of Definition 5, it leads to:

\[
V(x(0)) < \alpha(\lambda_{\text{max}}(\Pi) + h \cdot \lambda_{\text{max}}(Q)).
\]  

(31)

Consequently:

\[
V(x(k)) > x^T(k)E^TPEx(k) 
\geq \lambda_{\text{max}}(\Pi)x^T(k)E^TREx(k) \quad (32)
\]

Combining (29), (31) and (32) it can be obtained:

\[
\lambda_{\text{max}}(\Pi)x^T(k)E^TREx(k) < V(x(k))
\]

\[
< (\rho + 1)^{k} V(x(0)) < (\rho + 1)^{k} \alpha(\lambda_{\text{max}}(\Pi) + h \cdot \lambda_{\text{max}}(Q))
\]  

(33)

or:

\[
x^T(k)E^TREx(k) < (\rho + 1)^{k} \frac{\lambda_{\text{max}}(\Pi)}{\lambda_{\text{max}}(\Pi) + h \cdot \lambda_{\text{max}}(Q)}.
\]  

(34)

The condition (23) and the above inequality imply:

\[
x^T(k)E^TREx(k) < \beta. \quad \forall k \in K,
\]  

(35)

q.e.d.

**Remark 6:** It should be pointed out that the condition $E^TPE = E^T\Gamma^2\Pi \Gamma^2E$ in Theorem 4 is a natural consequence of Remark 2.

**Remark 7:** It should be noticed that the condition in Theorem 4 is not a typical LMI condition with respect to $\rho$, $P$ and $Q$.

However, it is possible to check that condition given (23) is guaranteed by imposing the conditions:

\[
\gamma_1 \beta < \rho \gamma_1,
\]

\[
0 < Q < \gamma_1,
\]

\[
\begin{bmatrix}
-\gamma_2 \beta (\rho + 1)^{k} & \gamma_2 \gamma_3 \sqrt{\alpha h} \\
\gamma_2 \sqrt{\alpha} & -\gamma_2 & 0 \\
\gamma_3 \sqrt{\alpha h} & 0 & -\gamma_3
\end{bmatrix} < 0
\]  

(36)

for some positive scalars $\gamma_1$, $\gamma_2$ and $\gamma_3$. Once $\rho$ is adopted properly, the conditions (21) and (22) can be turned into a LMI based feasibility problem.

V. Conclusion

Generally, this paper extends some of the basic results in the area of the non-Lyapunov stability to the particular class of linear discrete descriptor time delay systems.

To assure practical stability for LDDTDS it is not enough only to have the eigenvalues of matrix pair $(E, A)$ in the proper positions in the complex plane, but also to provide an impulse-free motion (compatible initial function) and certain conditions that needs to be fulfilled for the analyzed system. A new concept of the attractive practical stability has been introduced and applied to the particular class of autonomous LDDTDS. Furthermore, in Section 3, part of this result is a geometric counterpart of the algebraic theory in [5] supplemented with appropriate criteria to cover the need for system stability in the presence of actual time delay terms.

A new approach has been demonstrated to construct the non-Lyapunov stability theory based on some recent results given in [3] and in the LMI approach. A novel sufficient delay-dependent criterion for the finite time stability based on the LMI approach has been established as well.

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**References**


