A sublinear differential inclusion on strip-like domains

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Abstract—This paper deals with a sublinear differential inclusion problem \((P_{\lambda})\) depending on a parameter \(\lambda > 0\) which is defined on a strip-like domain subject to the zero Dirichlet boundary condition. By variational methods, we prove that for large values of \(\lambda\), problem \((P_{\lambda})\) has at least two non-zero axially symmetric weak solutions.

Key words and phrases: Eigenvalue problem; Strip-like domain; Multiple solutions; Axially symmetry; Differential inclusion.

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I. INTRODUCTION

We consider the following elliptic differential inclusion problem coupled with the zero Dirichlet boundary condition

\[
\begin{aligned}
-\Delta_{p} u + |u|^{p-2} u &\in \lambda \alpha(x,y) \partial F(u(x,y)) &\text{in} \ \Omega, \\
u & = 0 &\text{on} \ \partial \Omega, \\
\end{aligned}
\]

where \(\Omega = \omega \times \mathbb{R}^{N-m}, \omega \subset \mathbb{R}^{m}\) being bounded and open with smooth boundary, \(p > N, N - m \geq 2\), \(\Delta_{p}(u) = \text{div}(\nabla |\nabla u|^{p-2} \nabla u)\) is the \(p\)-Laplacian operator, \(\lambda\) is a positive parameter, \(\alpha \in L^{\infty}(\Omega) \cap L^{1}(\Omega)\) is an axially symmetric, nonnegative, nonzero function and \(\partial F\) stands for the generalized gradient of a locally Lipschitz function \(F : \mathbb{R} \rightarrow \mathbb{R}\).

The aforementioned problem has been studied not only from mathematical point of view but also for its applicability in mathematical physics (e.g., in the theory of fluid mechanics) where solutions of elliptic problems correspond to certain equilibrium state of the physical system. For similar problems, we refer the reader to the works [1], [4]-[7], [9]-[14], [17], [18], [24].

The motivation to consider this kind of inclusion comes from discontinuous phenomena. Namely, if \(f \in L_{\text{loc}}^{\infty}(\mathbb{R})\) is not necessarily continuous, the problem

\[
\begin{aligned}
-\Delta_{p} u + |u|^{p-2} u & = \lambda \alpha(x,y) f(u(x,y)) &\text{in} \ \Omega, \\
u & = 0 &\text{on} \ \partial \Omega, \\
\end{aligned}
\]

need not have a solution, due to the presence of certain gaps in the right hand side. However, if we replace the function \(f\) in \((P_{\lambda})\) by an interval \([f(-), f(\cdot)\]), where

\[
\begin{aligned}
f(s) & = \lim_{\delta \rightarrow 0^+} \text{essinf}_{|t-s|<\delta} f(t) \\
\overline{f}(s) & = \lim_{\delta \rightarrow 0^+} \text{esssup}_{|t-s|<\delta} f(t), \\
\end{aligned}
\]

the new set-valued problem may have solutions in a certain sense. Moreover, if \(F(s) = \int_{0}^{s} f(t) dt\) with \(f \in L_{\text{loc}}^{\infty}(\mathbb{R})\), then \(F\) is locally Lipschitz and \(\partial F(s) = [f(s), \overline{f}(s)]\) for every \(s \in \mathbb{R}\), see [15].

We study problem \((P_{\lambda})\) by using variational arguments which require certain compactness of embeddings. Usually, when we are dealing with bounded domains, Sobolev spaces can be compactly embedded into various Lebesgue spaces. However, when the domain is unbounded, no compactness can be expected of the Sobolev spaces due to dilations of translations. Consequently, in order to study our problem we need a compact embedding theorem which will exploit the symmetry of the strip-like domain, described recently in the paper [8]; namely, if \(p > N\), the subspace of axially symmetric functions of \(W_{0}^{1,p}(\Omega)\) is compactly embedded into \(L^{\infty}(\Omega)\). More precisely, the (closed) subspace of \(W_{0}^{1,p}(\Omega)\) consisting of the axially symmetric functions is

\[
W_{c}^{1,p}(\Omega) = \{ u \in W_{0}^{1,p}(\Omega) : u(x,\cdot) \text{ is radially symmetric for all } x \in \omega \}.
\]

As a result, we obtain some weak solutions with respect to the narrowed space \(W_{c}^{1,p}(\Omega)\). Note that \(W_{c}^{1,p}(\Omega)\) is a proper subspace of \(W_{0}^{1,p}(\Omega)\), thus further arguments are needed to prove that the solutions are actually weak solutions of the problem with respect to the whole space \(W_{0}^{1,p}(\Omega)\). The answer will be achieved by the so-called principle of symmetric criticality (see e.g. [21] for the smooth version). Recently, in [16], the authors extended this principle to locally Lipschitz functionals perturbed by a lower semicontinuous, proper and convex functional which will be useful in our investigations.
As we already pointed out, the main objective of our paper is to ensure the existence of solutions for the problem \((P_\lambda)\) where the natural functional framework is the Sobolev space \(W^{1,p}_0(\Omega)\).

In order to present our main result, we first recall that \(u \in W^{1,p}_0(\Omega)\) is a weak solution to problem \((P_\lambda)\) if for all \(v \in W^{1,p}_0(\Omega)\) there exists \(\xi_F \in \partial F(u(x,y))\) for a.e. \((x,y) \in \Omega\) such that
\[
\int_{\Omega} \left( |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2}uv \right) dx dy = \lambda \int_{\Omega} \alpha(x,y) \xi_F v(x,y) dx dy.
\]

In the sequel, we outline our approach and state the main result. We assume that the following hypotheses hold:
\[
\begin{align*}
(A) & \quad \inf_{\omega \times B(0,R)} \alpha(x,y) > 0, \text{ where } B(0,R) = \{ x \in \mathbb{R}^{N-m} : ||x||_{\mathbb{R}^{N-m}} < R \} \subset \mathbb{R}^{N-m}. \\
(F_1) & \quad \max_{|s| \to 0} \frac{\|\xi\|_{L^p(\Omega)}}{|s|^{p-1}} = 0; \\
(F_2) & \quad \lim_{|s| \to +\infty} \frac{F(s)}{|s|^{p-1}} = 0; \\
(F_3) & \quad F(s_0) > 0 \text{ for some } s_0 \in \mathbb{R}.
\end{align*}
\]

Our main result reads as follows:

**Theorem 1.1:** Assume that \(p > N \geq 2\), and let \(\Omega = \omega \times \mathbb{R}^{N-m}\), where \(\omega \subset \mathbb{R}^m\) is a bounded open domain with smooth boundary. Let \(\alpha \in L^\infty(\Omega) \cap L^1(\Omega)\) be a positive axially symmetric function satisfying (A), and let \(F : \mathbb{R} \to \mathbb{R}\) be a locally Lipschitz function with \(F(0) = 0\), satisfying \((F_1) - (F_3)\). Then there exists a \(\lambda_0 > 0\) such that for every \(\lambda > \lambda_0\) problem \((P_\lambda)\) has at least two axially symmetric non-zero, weak solutions in \(W^{1,p}_0(\Omega)\).

The proof of this theorem is based on variational arguments. To see this, we consider the functionals \(I, L : W^{1,p}_0(\Omega) \to \mathbb{R}\) defined by
\[
I(u) = \frac{1}{p} \|u\|^p, \quad L(u) = \int_{\Omega} \alpha(x,y) F(x,y) dx dy.
\]

Here, \(\|\cdot\|\) denotes the standard norm on \(W^{1,p}_0(\Omega)\). The energy functional associated with problem \((P_\lambda)\) is given by
\[
E_\lambda(u) = I(u) - \lambda L(u),
\]
which is a locally Lipschitz functional on \(W^{1,p}_0(\Omega)\). Furthermore, a standard argument shows that the critical points (in the sense of Chang) of \(E_\lambda\) are precisely the weak solutions of the problem \((P_\lambda)\). Moreover, due to the non-smooth principle of symmetric criticality of Palais (see [16]), the critical points of \(E_\lambda\) are the compact critical points of \(E_\lambda\) as well, so axially symmetric, weak solutions of the problem \((P_\lambda)\). Therefore, it is sufficient to guarantee critical points for \(E_\lambda\) in \(W^{1,p}_0(\Omega)\), where the compactness of the embedding \(W^{1,p}_0(\Omega) \to L^\infty(\Omega)\) will be deeply exploited.

**Remark 1.1:** Note that if the condition \((F_2)\) is replaced by
\[
(F_2)' \quad \lim_{|s| \to \infty} \max_{|s|\leq 1} \frac{\|\xi\|_{L^p(\Omega)}}{|s|^{p-1}} = 0,
\]
then for small values of \(\lambda > 0\), problem \((P_\lambda)\) has only the trivial solution. Indeed, if \(u \in W^{1,p}_0(\Omega)\) is a weak solution of \((P_\lambda)\), and we put as the test function \(v = u\) in relation (I.1), one obtains
\[
\int_{\Omega} \left( |\nabla u|^p + |u|^p \right) dx dy = \lambda \int_{\Omega} \alpha(x,y) \xi_F u dx dy \leq \lambda c_F \|\alpha\|_{L^\infty} \int_{\Omega} |u|^p dx dy,
\]
where \(c_F = \max_{s > 0} \max_{s \leq 1} \frac{\|\xi\|_{L^p(\Omega)}}{|s|^{p-1}} > 0\). Therefore, if \(\lambda < (c^\prime_F \|\alpha\|_{L^\infty})^{-1}\), then \(u = 0\).

**II. PROOF OF THEOREM 1.1**

Before proving our main result, we prove that our functional \(\mathcal{A}_\lambda\) is coercive and satisfies the non-smooth Palais-Smale condition on \(W^{1,p}_0(\Omega)\).

**Proposition 2.1:** The functional \(\mathcal{A}_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}\) is coercive for every \(\lambda > 0\).

**Proof:** Due to \((F_2)\), for every \(\varepsilon > 0\), there exists \(\delta(\varepsilon) > 0\) such that
\[
|F(t)| \leq \varepsilon|t|^p, \quad |t| > \delta(\varepsilon).
\]

Therefore, one can conclude that
\[
\mathcal{F}(u) = \int_{\{u(x,y) > \delta(\varepsilon)\}} \alpha(x,y) F(u(x,y)) dx dy + \int_{\{u(x,y) \leq \delta(\varepsilon)\}} \alpha(x,y) F(u(x,y)) dx dy \leq \varepsilon c^\prime_{\varepsilon} \|\alpha\|_{L^1} \|u\|^p + \|\alpha\|_{L^1} \sup_{|s| \leq \delta(\varepsilon)} |F(s)| = \varepsilon c^\prime_{\varepsilon} \|\alpha\|_{L^1} \|u\|^p + \|\alpha\|_{L^1} \sup_{|s| \leq \delta(\varepsilon)} |F(s)|,
\]
where \(c^\prime_\varepsilon\) is the embedding constant in \(W^{1,p}_0(\Omega) \to L^\infty(\Omega)\) (see Theorem 2.2 in [8]). Then, from (II.1), one can conclude, that
\[
\mathcal{A}_\lambda(u) = \left( \frac{1}{p} - \varepsilon \lambda \|\alpha\|_{L^1} c^\prime_{\varepsilon} \right) \|u\|^p - \lambda \sup_{|s| \leq \delta(\varepsilon)} |F(s)| > 0.
\]

In particular, if \(0 < \varepsilon < (p \lambda \|\alpha\|_{L^1} c^\prime_{\varepsilon})^{-1}\), then \(\mathcal{A}_\lambda(u) \to \infty\).

**Proposition 2.2:** For every \(\lambda > 0\), \(\mathcal{A}_\lambda\) satisfies the non-smooth PS-condition.

**Proof:** Let \(\lambda > 0\) be fixed. We consider a Palais-Smale sequence \(\{u_n\} \subset W^{1,p}_0(\Omega)\) for \(\mathcal{A}_\lambda\), i.e., \(\mathcal{A}_\lambda(u_n) \to \mathcal{A}_\lambda(u) \geq -\varepsilon_n \|u - u_n\|\) for some \(\varepsilon_n \to 0^+\) and \(\{\mathcal{A}_\lambda(u_n)\}\) bounded in \(W^{1,p}_0(\Omega)\). Since \(\mathcal{A}_\lambda\) is coercive, the sequence \(\{u_n\}\) is bounded. Therefore taking a subsequence if necessary, we may assume that \(u_n \rightharpoonup u\) weakly in \(W^{1,p}_0(\Omega)\) and \(u_n \to u\) strongly in \(L^\infty\) (note that \(W^{1,p}_0(\Omega) \to L^\infty(\Omega)\) is compact, see Theorem 2.2 in [8]). We clearly have that
\[
\mathcal{I}(u_n) - \mathcal{I}(u) + \left( \int_{\Omega} \int (|\nabla u_n|^2 - |\nabla u|^2) \right) = \int_{\Omega} \left( |\nabla u_n|^2 - |\nabla u|^2 \right) dxdy.
\]
where means that $u_n \to u$ strongly in $W^{1,p}_c(\Omega)$.  

\textbf{Proof of Theorem 1.1.} By assumption (A), we have that there exists $B(x_0, r) \subset \omega$ such that
\[
\inf_{B(x_0, r) \times B(0, R)} \alpha(x, y) \geq \inf_{\omega \times B(0, R)} \alpha(x, y) > 0.
\]
Let $K := B(x_0, r) \times B(0, R)$ and let $\tau > 0$ such that $K_\tau := B(x_0, r + \tau) \times B(0, R + \tau) \subset \Omega$. We can construct a function $u_\tau \in W^{1,p}_c$ such that $u_\tau \mid_{\Omega \setminus K_\tau} = 0$, $u_\tau \mid_{K_\tau} = s_0$ and $\|u_\tau\|_{L^\infty} \leq |s_0|$. Then
\[
\mathcal{F}(u_\tau) = \int_\Omega \alpha(x, y) F(u_\tau(x, y)) \, dx \, dy = \int_\Omega \alpha(x, y) F(u_\tau(x, y)) \, dx \, dy + \int_{K_\tau} \alpha(x, y) F(u_\tau(x, y)) \, dx \, dy \\
\geq \int_K \alpha(x, y) F(u_\tau(x, y)) \, dx \, dy + \int_{K_\tau} \alpha(x, y) F(u_\tau(x, y)) \, dx \, dy \\
\geq F(s_0) \text{vol}(K) \cdot \inf_K \alpha - \|\alpha\|_{L^\infty} \text{vol}(K_\tau \setminus K) \max_{s \in [-s_0, s_0]} F(s).
\]
From the fact that $F(s_0) \text{vol}(K_\tau \setminus K)$ tends to 0 whenever $\tau \to 0$, one can choose $\tau_0$ small enough such that $\mathcal{F}(u_{\tau_0}) > 0$. Now let $u_1 = u_{\tau_0}$; then $A = \frac{1}{p} \|u_1\|^p > 0$ and $B = \mathcal{F}(u_1) > 0$. It follows that there exists a $\lambda_0$ such that, for every $\lambda \geq \lambda_0$,
\[
\mathcal{A}(u_1) < 0.
\]
Now, fix $\lambda > \lambda_0$. From Proposition 2.1 and Proposition 2.2 we can conclude that $\mathcal{A}(u_1)$ has a global minimum with negative energy level, namely $u^*_\lambda$,
\[
\mathcal{A}(u^*_\lambda) \leq \mathcal{A}(u_1) < 0 = \mathcal{A}(0),
\]
therefore $u^*_\lambda \neq 0$.

From (F1) it follows that for fixed $\frac{1}{p\lambda e} \|\alpha\|_{L^1} > \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that
\[
\max\{\|\xi\| : \xi \in \partial F(u_\tau(x, y))\} \leq \varepsilon |s|^{p-1}, |s| < \delta,
\]
therefore for every $\xi \in \partial F(s)$, $|s| \leq \delta$ one has,
\[
|\xi| \leq \varepsilon \cdot |s|^{p-1}.
\]
Using the Lebourg’s mean value theorem (see Theorem 2.3.7, [3]), we obtain
\[
|F(s)| = |F(s) - F(0)| \leq |\xi_\theta s|
\]
for some $\xi_\theta \in \partial F(\theta s)$, $\theta \in (0, 1)$, which means that
\[
|F(s)| \leq \varepsilon \cdot |s|^p \text{ if } |s| < \delta.
\]
Thus, if \( u \in W^{1,p}_c(\Omega) \) with \( \|u\| = \rho < \min \left\{ \frac{\delta}{\varepsilon}, \|u_1\| \right\} \), then
\[
\mathcal{A}_\lambda(u) = \frac{1}{p} \|u\|^p - \lambda F(u) \\
\geq \frac{1}{p} \|u\|^p - \varepsilon \lambda \varepsilon^p \|\alpha\|_{L^1} \|u\|^p \\
= \|u\|^p \left( \frac{1}{p} - \varepsilon \lambda \varepsilon^p \|\alpha\|_{L^1} \right) \\
= \rho^p \left( \frac{1}{p} - \varepsilon \lambda \varepsilon^p \|\alpha\|_{L^1} \right) > 0.
\]
Since
\[
\inf_{\|u\| = \rho} \mathcal{A}_\lambda(u) > 0 = \mathcal{A}_\lambda(0) > \mathcal{A}_\lambda(u_1),
\]
and \( \mathcal{A}_\lambda \) satisfies the non-smooth PS-condition, we are in the position to apply the mountain pass theorem for locally Lipschitz functionals (see Motreanu and Panagiotopoulos [20] or Pucci and Radulescu [22]). Consequently, the functional \( \mathcal{A}_\lambda \) has a critical point with positive energy level, whenever \( \lambda > \lambda_0 \), namely \( u_1^\lambda \in W^{1,p}_c(\Omega) \), i.e.,
\[
\mathcal{A}_\lambda(u_1^\lambda) > 0 = \mathcal{A}_\lambda(0),
\]
which means that \( u_1^\lambda \neq 0 \). Therefore, for \( \lambda > \lambda_0 \) the functional \( \mathcal{A}_\lambda \) has at least two distinct critical points, \( 0 \neq u_1^\lambda \neq u_1^{\lambda_1} \).

Let \( G = id_{\mathbb{R}^{N-m}} \times O(N-m) \) be the group acting on \( W^{1,p}_0(\Omega) \) in the standard manner, i.e., for every \( g = id_{\mathbb{R}^{N-m}} \times g_0 \in G \) with \( g_0 \in O(N-m) \), \( u \in W^{1,p}_0(\Omega) \) and \( (x,y) \in \Omega = \omega \times \mathbb{R}^{N-m}, \)
\[
(gu)(x,y) = u(x,g_0^{-1}y).
\]
By this action, the compact group \( G \) acts isometrically on \( W^{1,p}_0(\Omega) \). Moreover, since \( \alpha \) is axially symmetric, the functional \( \mathcal{E}_\lambda \) is \( G \)-invariant. On account of the non-smooth principle of symmetric criticality (see [16]), the critical points of \( \mathcal{A}_\lambda \) are actually critical points for the original energy functional \( \mathcal{E}_\lambda \), thus weak solutions of the problem \((P_\lambda)\), which are axially symmetric.

**Remark 2.1:** (a) By using a recent result of Ricceri [23], we can prove that the number of solutions for the problem \((P_\lambda)\) is stable with respect to any small perturbation of the right hand side.

(b) If \( F \) is even, we can improve the number of solutions by using group-theoretical arguments when \( N - m \) is large enough. More precisely, for \( N - m \geq 2 \) and \( i \in \{1, \ldots, \lfloor \frac{N-m}{2} \rfloor \} \), let \( G_{N-m,i} = id_{\mathbb{R}^{N-m}} \times O(N-m) \times O(N-m) \) if \( N - m = 2i \) and \( G_{N-m,i} = id_{\mathbb{R}^{N-m}} \times O(i) \times O(N-m-2i) \times O(i) \) if \( N - m \neq 2i \), and \( \tau_i \) its corresponding involution function, i.e.,
\[
\tau_i = id_{\mathbb{R}^{N-m}} \times \left( \begin{array}{cc} 0 & \frac{1}{i} \frac{N-m}{2} \frac{1}{i} \frac{N-m}{2} \\ \frac{1}{i} \frac{N-m}{2} & 0 \end{array} \right) \quad \text{if} \quad N - m = 2i,
\]
and
\[
\zeta_i = id_{\mathbb{R}^{N-m}} \times \left( \begin{array}{ccc} 0 & 0 & \frac{1}{i} \frac{N-m}{2} \\ 0 & 0 & \frac{1}{i} \frac{N-m}{2} \\ \frac{1}{i} \frac{N-m}{2} & 0 & 0 \end{array} \right) \quad \text{if} \quad N - m \neq 2i.
\]
Let \( \tilde{G}_{N-m,i} \) be the group generated by \( G_{N-m,i} \) and \( \tau_i \), i.e., \( \tilde{G}_{N-m,i} = [G_{N-m,i}; \tau_i] = G_{N-m,i} \cup \tau_i G_{N-m,i}. \) Note that only two types of elements can be found in \( \tilde{G}_{N-m,i} \), namely, elements of the form \( G_{N-m,i} \) and \( \tau_i G_{N-m,i} \). The action of the group \( \tilde{G}_{N-m,i} \) on \( W^{1,p}_0(\Omega) \) is defined by
\[
(gu)(x,y) = \begin{cases} 
\begin{array}{ll}
\begin{cases} u(x,g_0^{-1}y) & \text{if} \quad g \in G_{N-m,i}; \\
-u(x,\tau_i g_0^{-1}y) & \text{if} \quad g \in G_{N-m,i} \setminus G_{N-m,i}.
\end{cases}
\end{cases}
\end{cases}
\]
for \( g = id_{\mathbb{R}^{N-m}} \times g_0 \in G_{N-m,i}, u \in W^{1,p}_0(\Omega) \) and \( (x,y) \in \Omega \). Now, let
\[
W^{1,p}_0(\Omega) = \{ u \in W^{1,p}_0(\Omega) : gu = u, \forall g \in \tilde{G}_{N-m,i} \}.
\]
First, note that these spaces are compactly embedded into \( L^\infty(\Omega), \ W^{1,p}_0(\Omega) \cap W^{1,p}_0(\Omega) = \{0\} \) and \( W^{1,p}_0(\Omega) \cap W^{1,p}_0(\Omega) = \{0\} \) for every \( i \neq j \). Moreover, the energy functional \( \mathcal{E}_\lambda \) is \( \tilde{G}_{N-m,i} \)-invariant. Consequently, instead of \( W^{1,p}_0(\Omega) \), we may apply the above machinery to the spaces \( W^{1,p}_i(\Omega), i \in \{1, \ldots, \lfloor \frac{N-m}{2} \rfloor \} \), obtaining at least 2 \( \lfloor \frac{N-m}{2} \rfloor \) distinct pairs of nonzero weak solutions for \((P_\lambda)\) in \( W^{1,p}_0(\Omega) \).

**Remark 2.2:** By using genetic algorithms we are planning to locate approximate solutions to our differential inclusion system \((P_\lambda)\) in the spirit of the paper of Mellal et al. [19].

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