An Approach to Design of Sliding Mode Based Generalized Predictive Control

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Abstract—This paper presents the combination of generalized predictive and sliding mode control techniques in order to improve the system robustness to parameter variation. The proposed control algorithm belongs to the group of chattering-free sliding mode control laws, and it provides the minimum value of the cost function in the presence of parameter perturbations. Digital simulation results are given to verify the sliding mode based generalized predictive controller.

I. INTRODUCTION

Model predictive control (MPC) may be seen as a further development of the linear optimal control developed in the 50’s and 60’s [1], [2]. Traditionally, MPC has been applied to systems with relatively slow dynamics, in particular those found in the chemical processing industries. However, more recently this control method has found application in a wider range of industries, owing both to more efficient optimization formulations and the availability of computational power. While MPC may offer great flexibility in formulating the control problem and the choice of system model, it can in some cases lack robustness, i.e., it can be highly dependent on model quality and knowledge of disturbances. The use of sliding mode control (SMC) to cope with the robustness problem of MPC seems to be a logical choice.

SMC represents a special class of variable structure control [3], [4], [5]. This nonlinear control provides system robustness to parameter perturbations and external disturbances. A sliding mode exists in system when a system state is forced, by discontinuous control with infinite switching, to move along a predefined sliding surface. In modern implementation of SMC, using digital signal processors, SMC is subjected to discretization. This results in a quasi-sliding motion [6] and generates an undesirable chattering phenomenon in the vicinity of a sliding surface. The overview of existing digital SMC systems is presented in [7]. The special subclass of digital SMC algorithms, based on input/output model, combines the generalized minimum variance control (GMVC) and digital SMC techniques [8], [9], [10], [11], [12]. This led to the use of generalized predictive control (GPC), instead of GMVC, to provide better system characteristics. GPC is a particular subclass of MPC, realized by using only input/output time sequences. As GMVC, GPC should replace the equivalent control [13] in traditional digital SMC design. This combination of SMC and GPC is analyzed in [14], [15], [16], [17].

In this paper, we propose the modification of SMC component of the solution presented in [14], to improve the system robustness. The minimum value of the cost function for perturbed systems is ensured by using the chattering-free SMC algorithm.

The paper is organized as follows. In Section II, the control problem is introduced. Section III briefly describes GPC. The proposed sliding mode based GPC is presented in Section IV. The existence condition of sliding mode is thoroughly discussed herein. Section V presents the results of digital simulation of the system with the proposed control algorithm. Section VI contains some concluding remarks.

II. PROBLEM STATEMENT

We consider the discrete-time model of single-input-single-output plant [14]:

\[ A(z^{-1})y_k = z^{-1}B(z^{-1})u_k , \] (1)

with:

\[ A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n} , \] (2)

\[ B(z^{-1}) = b_0 + b_1 z^{-1} + \ldots + b_n z^{-n} , \] (3)

where \( z^{-1} \) denotes the unit delay operator, \( u_k \) and \( y_k \) are the input and the output of plant, \( n_u \) and \( n_y \) are the degrees of the polynomials \( A(z^{-1}) \) and \( B(z^{-1}) \), respectively. We assume that the control input \( u_k \) is bounded, i.e. \( |u_k| \leq \bar{U} \) and \( \bar{U}=\text{const.} \), which typically holds in practical implementations, due to saturation nonlinearities existing in real plants.

The goal of design is to find the control law which will minimize the cost function:

\[ J = s^T s + \bar{u}^T \lambda \bar{u} , \] (4)

where \( \lambda \) is a control weighting constant and:

\[ s = [s_{k-1} \ldots s_{k+N-1}]^T , \] (5)

\[ \bar{u} = [\Delta u_k \ldots \Delta u_{k+N-1}]^T , \] (6)

with \( N \) denoting prediction horizon, \( \Delta = 1 - z^{-1} \) denoting the difference operator, and:

\[ s_{k+j} = C(z^{-1})(y_{k+j} - r_{k+j}) + Q(z^{-1})\Delta u_{k+j}, \ j = 1, \ldots , N , \] (7)
representing the variable whose minimum value ensures the good system tracking of the reference input \( r_k \) i.e. in the ideal case the zero value of the tracking error:

\[ e_1 = y_k - r_k , \quad (8) \]

The polynomials \( C(z^{-1}) \) and \( Q(z^{-1}) \) are given by:

\[ C(z^{-1}) = c_0 + c_1 z^{-1} + \ldots + c_n z^{-n} , \quad (9) \]
\[ Q(z^{-1}) = q_0 + q_1 z^{-1} + \ldots + q_n z^{-n} , \quad (10) \]

having the degrees \( n_c \) and \( n_q \), respectively, and should be selected to assign the desired closed-loop systems dynamics. The variable \( \hat{u}_k \) is obtained from the control input as:

\[ \hat{u}_k = C(z^{-1})u_k , \quad (11) \]

### III. GENERALIZED PREDICTIVE CONTROL

In order to realize GPC, let us consider the next two Diophantine equations [14]:

\[ E_j(z^{-1})A(z^{-1})\Delta + z^{-1}F_j(z^{-1}) = 1 , \quad (12) \]
\[ E_j(z^{-1})B(z^{-1}) = G_j(z^{-1}) + z^{-1}H_j(z^{-1}) , \quad (13) \]

whose solutions are the following polynomials:

\[ E_j(z^{-1}) = e_0 j + e_1 j^{(1)} + \ldots + e_{j-1} j^{(i-1)} , \quad (14) \]
\[ F_j(z^{-1}) = f_0 j + f_1 j^{(1)} + \ldots + f_{j-1} j^{(i-1)} , \quad (15) \]
\[ G_j(z^{-1}) = g_0 j + g_1 j^{(1)} + \ldots + g_{j-1} j^{(i-1)} , \quad (16) \]
\[ H_j(z^{-1}) = h_0 j + h_1 j^{(1)} + \ldots + h_{j-1} j^{(i-1)} , \quad (17) \]

for \( j = 1, \ldots, N \). The first Diophantine equation (12) is used for obtaining the prediction output, whereas the second one (13) distinguishes future and past control values.

By multiplying the both sides of (1) with \( E_j(z^{-1})\Delta \) and by substituting (12) and (13) in the obtained result, the variable (7) can be written as:

\[ s_j = C(z^{-1})[F_j(z^{-1})y_j + G_j(z^{-1})\Delta u_{j+1} + H_j(z^{-1})\Delta u_{j-a}]
- C(z^{-1})r_{j+1} + Q(z^{-1})\Delta u_{j-a} , \quad (18) \]

for \( j = 1, \ldots, N \), or in more compact form as:

\[ s = C(z^{-1})F(z^{-1})y_j + G\hat{u} - C(z^{-1})r + [C(z^{-1})H(z^{-1}) + Q(z^{-1})]\Delta u_{j-a} , \quad (19) \]

where:

\[ F(z^{-1}) = [F_1(z^{-1}) \ldots F_N(z^{-1})]^T , \quad (20) \]
\[ Q(z^{-1}) = Q(z^{-1})I , \quad (21) \]
\[ H(z^{-1}) = [H_1(z^{-1}) \ldots H_N(z^{-1})]^T , \quad (22) \]
\[ u = [\Delta u_1 \ldots \Delta u_{k-N-1}]^T , \quad (23) \]
\[ r = [r_{k+1} \ldots r_{k+a}]^T , \quad (24) \]

The next step is to minimize the cost function (4) with respect to control \( \hat{u}_1 \), taking into account (19), which gives the following control vector:

\[ \hat{u}_1 = -(G^T(G + \lambda I))^{-1}G^T[C(z^{-1})F(z^{-1})y_j - C(z^{-1})r + \lambda[C(z^{-1})H(z^{-1}) + Q(z^{-1})]\Delta u_{j-a}] , \quad (25) \]

GPC algorithm is now given by the first row of (26):

\[ \Delta u_{j+1} = -m_1(C(z^{-1})F(z^{-1})y_j - C(z^{-1})r + \lambda[C(z^{-1})H(z^{-1}) + Q(z^{-1})]\Delta u_{j-a}] \]

where:

\[ M = (G^T(G + \lambda I))^{-1}G^T[m_1, m_2, \ldots, m_N]^T \quad (26) \]

GPC can now be rewritten as [14]:

\[ \Delta u_{j+1} = -\Phi^{-1}(z^{-1})(Y(z^{-1})y_j - \Psi(z^{-1})r_{j+a}) , \quad (29) \]

with:

\[ \Phi(z^{-1}) = C(z^{-1}) + m_1(z^{-1})[C(z^{-1})H(z^{-1}) + Q(z^{-1})] \]
\[ \Psi(z^{-1}) = m_1C(z^{-1})F(z^{-1}) \]

The substitution of (29) in (1) yields the closed-loop system transfer function described by:

\[ y_{k+1} = \frac{B(z^{-1})\Psi(z^{-1})}{\Phi(z^{-1})A(z^{-1}) + B(z^{-1})Y(z^{-1})}r_{k+a} \quad (30) \]

As one can see, by the proper choice of the polynomials \( C(z^{-1}) \) and \( Q(z^{-1}) \), the desired closed-loop dynamics can be obtained. Moreover, the polynomials \( C(z^{-1}) \) and \( Q(z^{-1}) \) can be calculated uniquely from:

\[ \Phi(z^{-1})A(z^{-1}) + z^{-1}B(z^{-1})Y(z^{-1}) = P(z^{-1}) \quad (31) \]

if \( n_c = n_b + 1 \) and \( n_q = n_b + n_c - 1 \) [14], where \( P(z^{-1}) \) is the polynomial obtained by assigning the desired closed-loop poles. Furthermore, the tracking error disappears if:

\[ Y(1) = \Psi(1) \quad (32) \]

### IV. SLIDING MODE BASED GENERALIZED PREDICTIVE CONTROL

The basic idea of existing GPC strategies with sliding mode is to use GPC as a replacement for the so-called equivalent control in variable structure systems with sliding mode [7], [13]. In [14], the authors propose the sliding surface in the form of:

\[ \sigma_{k+1} = \Phi(z^{-1})\Delta u_k + Y(z^{-1})y_k - \Psi(z^{-1})r_{k+a} = 0 \quad (33) \]

It is obvious that in sliding mode \( \sigma_{k+1} = 0 \), the equivalent control \( \Delta u_{k+1}^{eq} \) corresponds to GPC given by (29):

\[ \Delta u_{k+1}^{eq} = \Delta u_{k+1}^{GPC} \quad (34) \]
That means if sliding mode exists, the minimum of the cost function (4) is ensured for the nominal plant model. In the case of parameter perturbations, an additional control term should be introduced in the control algorithm to ensure the existence of sliding mode.

Unlike the solutions given in [14], we proposed the switching function of SMC component as:
\[ \dot{\hat{\sigma}}_{k+1} = Y(z^{-1})y_k - \Psi(z^{-1})r_{k+N}. \] (38)

Note that:
\[ \hat{\sigma}_{k+1} = 0, \] (39)
defines the sliding surface in our case, and:
\[ \sigma_{k+1} = \hat{\sigma}_{k+1} + \Phi(z^{-1})\Delta u_k. \] (40)

We should determine the control law now, which will establish the sliding motion in the vicinity of (39) and provide \( \sigma_{k+1} = 0 \) at the same time. This sliding mode based control algorithm can be expressed now in the following form:
\[ \Delta u_k = -\Phi^{-1}(z^{-1})\{Y(z^{-1})y_k - \Psi(z^{-1})r_{k+N} - \dot{\hat{\sigma}}_k + \min(|\dot{\hat{\sigma}}|, \alpha)\text{sgn}(\dot{\hat{\sigma}})\}. \] (41)

**Theorem 1.** The system (1) with the control law (41) guarantees \( \sigma_{k+1} = 0 \) and vanishing of the tracking error if \( \alpha > 3\Lambda = \max \{\Phi(z^{-1})\Delta u_{k+1}\} \). (42)

*Proof.* As the input signal is bound by assumption, there is a constant \( \alpha \) always satisfying (42). The insertion of (41) in (40), taking into account (38), yields:
\[ \sigma_{k+1} = \dot{\hat{\sigma}}_k - \min(|\dot{\hat{\sigma}}|, \alpha)\text{sgn}(\dot{\hat{\sigma}}), \] (43)

i.e.
\[ \sigma_{k+1} = \dot{\hat{\sigma}}_k - \min(|\dot{\hat{\sigma}}|, \alpha)\text{sgn}(\dot{\hat{\sigma}}) - \Phi(z^{-1})\Delta u_{k+1}. \] (44)

Suppose that \( \sigma_1 > 0, \dot{\hat{\sigma}}_1 > 0 \) and \( \dot{\hat{\sigma}}_1 > \alpha \). Then, \( \sigma_1 \) converges to the domain:
\[ \Sigma = \{ \sigma_1 : |\sigma_1| < \alpha + \Lambda \}, \] (45)
and enters it at \( k = K_0 \) determined by:
\[ K_0 = \text{int} \{ (|\sigma_0| - \alpha - \Lambda)/(\alpha - \Lambda) \} + 1, \] (46)
where \( \sigma_0 \) denotes the initial value of (40). Let \( \sigma_1 = \alpha + \Lambda \), then \( \dot{\hat{\sigma}}_1 \) could be still greater than \( \alpha \), but in the next step \( \dot{\hat{\sigma}}_{k+1} < 3\Lambda < \alpha \). It means that immediately after the entrance in the domain (45), \( \dot{\hat{\sigma}}_1 < \alpha \) and (44) becomes:
\[ \sigma_{k+1} = \sigma_1 - \dot{\hat{\sigma}}_1 - \Phi(z^{-1})\Delta u_{k+1} = 0. \] (47)

Assume that \( \dot{\hat{\sigma}} > 0 \) and \( \sigma < 0 \). In that case, \( \Phi(z^{-1})\Delta u_{k+1} \) is negative and \( \dot{\hat{\sigma}} < |\Phi(z^{-1})\Delta u_{k+1}| < \alpha \) so (47) occurs. The proof is similar for \( \dot{\hat{\sigma}} < 0 \). □

**Corollary 1.** The control law, defined by (41) and (42), ensure the existence of quasi-sliding mode in (1) in the vicinity of sliding surface \( \dot{\hat{\sigma}}_{k+1} = 0 \).

*Proof.* The implementation of (41) in (40) gives:
\[ \dot{\hat{\sigma}}_{k+1} = \dot{\hat{\sigma}}_k - \min(|\dot{\hat{\sigma}}|, \alpha)\text{sgn}(\dot{\hat{\sigma}}) - \Phi(z^{-1})\Delta u_k. \] (48)

If \( \alpha \) is selected in accordance with (42), \( \dot{\hat{\sigma}}_k \) reaches the domain:
\[ \dot{\Sigma} = \{ \dot{\hat{\sigma}}_1 : |\dot{\hat{\sigma}}_1| < \alpha + \Lambda \}, \] (49)
in \( K_0 \) sampling periods:
\[ \dot{\Sigma} = \text{int} \{ (|\dot{\sigma}_0| - \alpha - \Lambda)/(\alpha - \Lambda) \} + 1, \] (50)
with \( \dot{\sigma}_0 \) representing the initial value of (38). In (49) \( \dot{\sigma}_1 < \alpha \), so \( \dot{\hat{\sigma}}_{k+1} = -\Phi(z^{-1})\Delta u_k \) and
\[ \dot{\hat{\sigma}}_{k+1} < \Lambda. \] (51)
for every \( k \) □

The Corollary 1 presents the results of chattering-free digital SMC [18], obtained by discretization process of the well-known chattering-free power rate reaching law method [19]. In this paper, we have got the similar chattering-free control law, trying to ensure \( \sigma_{k+1} = 0 \) by using the switching function \( \dot{\hat{\sigma}}_k \) of SMC, defined by (38).

V. DIGITAL SIMULATION RESULTS

In order to verify the proposed sliding mode based generalized predictive controller, the digital simulations are performed by using the Van der Vusse reactor as a plant. This chemical process is very often utilized as a benchmark problem for design of process control algorithms. The example and the parameter values of the reactor are taken from [14] and [20]. The normalized model of process is given by:
\[ \dot{x}_1 = -50x_1 - 10x_2^2 + u(10 - x_1), \] (52)
\[ \dot{x}_2 = 50x_1 - 100x_2 + u(-x_2), \] (53)
\[ y = x_2, \] (54)
where \( x_1 \) and \( x_2 \) denote the concentrations of components A and B respectively, and \( u \) is an inlet flow rate. The operating point is determined by \( X_{10} = 3.0 \), \( X_{20} = 1.12 \) and \( U_0 = 34.3 \).

To obtain the discrete-time plant model (1), the linearization of nonlinear process (52)-(54) has been done around the operating point, and, then, the discretization has been performed with the sampling period \( T = 0.005 \).h. The results are the polynomials:
\[ A(z^{-1}) = 1 - 0.997z^{-1} + 0.248z^{-2}, \] (55)
\[ B(z^{-1}) = -1.29 \cdot 10^{-3} + 3.73 \cdot 10^{-3} z^{-1}. \] (56)

The zero initial conditions of the plant are taken and the polynomials \( C(z^{-1}) \) and \( Q(z^{-1}) \) are chosen as:
\[ C(z^{-1}) = 1 + 151.667z^{-1} - 143.502z^{-2} + 33.675z^{-3}, \] (57)
\[ Q(z^{-1}) = -879.661 + 828.859z^{-1} - 194.331z^{-2} + 0.012z^{-3}, \] (58)
yielding the following the closed-loop poles: \( p_1 = 0.1, \) \( p_2 = 0.2, \) \( p_3 = 0.3, \) \( p_4 = 0.4, \) \( p_5 = 0.5, \) \( p_6 = 0.6, \) and \( p_7 = 0.7. \) The prediction horizon of the proposed controller is \( N = 5 \) and \( \alpha = 1. \) The reference input is \( r_t = 0.62 \) for \( k = 1, 2, \ldots, \) and the system starts from the
steady state corresponding to the initial reference $r_0 = X_{20} = 1.12$

Fig. 1 presents the output signal of the Van der Vusse reactor with GPC. The system response is not satisfactory since the coefficients of polynomials $A(z^{-1})$ and $B(z^{-1})$ significantly differs from the ‘nominal’ values used in the calculation of GPC parameters. The control (29) is shown in Fig.2.

The implementation of the proposed sliding mode based GPC (SMGPC) in the control of Van der Vusse reaction process yields the results presented in Figs. 3-6. The output response is given in Fig. 3, and the system response is much better comparing to the response of system with GPC. The control signal (41) is depicted in Fig. 4. As one can see, there is no high frequency component in the control signal, so there is no chattering phenomenon at all. The zero value of $\sigma_i$ is ensured as it is shown in Fig. 5. The quasi-sliding mode is reached on sliding surface defined by the switching function $\hat{\sigma}_i$ (Fig. 6).

In order to analyze the robustness of the proposed SMGPC, 20% perturbation of the plant parameters is considered in digital simulation. Figs. 7-10 show the responses of the perturbed system with the proposed control. The variations of plant parameters cause the oscillation in output response (Fig. 7). There is no chattering in the system, as the control contains only low frequency components (Fig. 8). The zero value of $\sigma_i$ is still ensured (see Fig. 9), whereas the quasi-sliding motion occurs in the vicinity of $\hat{\sigma}_i = 0$, determined by (51).
The novel sliding mode based generalized predictive control is presented in this paper. The advantages of both sliding mode control and generalized predictive control are combined in order to obtain the control algorithm that would improve the robustness of system to parameter perturbations. The given control law ensures the cost function minimum in the presence of internal disturbances. As the main advantage of model predictive control, compared to classical optimal control, is the ability to handle constraints in both inputs and states/outputs, the future work should be consider the inclusion of constraints in generalized predictive control formulation.

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