Chebyshev Type Inequalities for Pseudo-Integrals of Set-Valued Functions

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Abstract—Chebyshev type inequalities for pseudo-integrals of set-valued functions for two classes of semirings are shown. One of them is the case when pseudo-integral of set-valued function is based on a g-semiring, where g is an increasing and continuous function, and the other case is when semiring is \([a, b], \max, \circ\), where \(\circ\) is generated by an increasing and continuous generator g.

Keywords. Chebyshev inequality, pseudo-operations, pseudo-integral, set-valued functions, set-valued pseudo-integral.

I. INTRODUCTION

Classical Chebyshev integral inequality is widely used in applied mathematics in the areas such as: economics, finance, decision making, etc. Agahi, Mestiar and Ouyang [1] proved Chebyshev type inequalities for two classes of pseudo-integral of comonotone functions. Pap and Štrboja (see [12]) also proved the Chebyshev type inequality for the pseudo integral for two characteristic cases: one, when pseudo-operations are defined by monotone and continuous function g, and two, when they considered the semiring \([a, b], \max, \circ\), where \(\circ\) is generated. In [12] measurable functions are of the same monotonicity.

Set-valued functions are a useful analytical tool in several practical areas, specially in the economic analysis (problems of individual demand, mean demand, competitive equilibrium, coalition production economies, etc.) (see [7]). The integration of set-valued functions has roots in the Aumann’s research based on the classical Lebesgue integral, [2]. Some generalizations of this approach that relied on an extension of the Lebesgue integral known as \(L\)-integral (see [14]) and on the Choquet integral have been investigated in [16] and [6], respectively. Also, the Sugeno type of the integration of set-valued functions as well as further generalizations based on generalized fuzzy integrals have been presented in [15].

The field of the pseudo-analysis has been chosen for the background of this paper, since it presents a contemporary mathematical theory that is being successfully applied in many different areas of mathematics as well as in various practical problems (see [8], [11], [13]). An approach to the problem of the integration of set-valued functions from the pseudo-analysis’ point of view has been proposed in [3], [4] and [5]. The integral introduced in [3] is based on the pseudo-integral (see [11]), i.e., pseudo-analysis’ counterpart of the classical integral.

The paper is organized as follows. The pseudo-operations, pseudo-integral and Chebyshev integral inequality for pseudo-integral are given in the second section. The third section contains the generalization of the pseudo-integral done in the Aumann’s style ([2], [3], [5]). In Section IV the generalization of the Chebyshev integral inequality is proved for the set-valued function for two characteristic cases: one, when pseudo-operations are defined by monotone and continuous function g, and two, when the semiring \([a, b], \max, \circ\), where \(\circ\) is generated, is considered.

II. PRELIMINARY NOTIONS

The basic preliminary notions needed in this paper are the notions of the pseudo-operations and the semiring.

Let \([a, b]\) be a closed subinterval of \([-\infty, +\infty]\) (in some cases semiclosed subintervals will be considered) and let \(\preceq\) be a total order on \([a, b]\).

Pseudo-addition is a function \(\oplus : [a, b] \times [a, b] \rightarrow [a, b]\) which is commutative, non-decreasing (with respect to \(\preceq\)), associative and with a zero element, denoted by \(0\).

Pseudo-multiplication is a function \(\circ : [a, b] \times [a, b] \rightarrow [a, b]\) which is commutative, positively non-decreasing \((x \preceq y, z \in [a, b], x \circ z \preceq y \circ z)\), associative and for which there exists a unit element denoted by 1.

A semiring is a structure \([(a, b), \oplus, \circ]\) such that the following holds:

- \(\oplus\) is a pseudo-addition;
- \(\circ\) is a pseudo-multiplication;
- \(0 \circ x = 0\);
- \(x \circ (y + z) = (x \circ y) \oplus (x \circ z)\).

There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and non idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by the monotone and continuous generator function \(g : [a, b] \rightarrow [0, +\infty]\), i.e. g-semirings, form the second class, and semirings with both idempotent operations belong to the third class ([18], [9], [11]).
Total order \( \preceq \) is closely connected to the choice of the pseudo-addition. If \( \oplus \) is an idempotent operation (semirings of the first and the third class), the total order is induced in the following manner
\[
x \preceq y \text{ if and only if } x \oplus y = y,
\]
and if \( (\{a, b\}, \oplus, \odot) \) is a semiring of the second class given by a generator \( g \), total order is given by
\[
x \preceq y \text{ if and only if } g(x) \leq g(y).
\]
In all three cases the strict order \( < \) has the following form:
\[
x < y \text{ if and only if } x \preceq y \text{ and } x \neq y.
\]
Let \( (\{a, b\}, \oplus, \odot) \) be a semiring. It is supposed that \( (\{a, b\}, \oplus) \) and \( (\{a, b\}, \odot) \) are complete lattice ordered semigroups. It is supposed that the \( \{a, b\} \) is endowed with the metric \( d \) which is compatible with \( \sup \) and \( \inf \) and satisfies at least one of the conditions:
\[i) \ d(x_1 \oplus y_1, x_2 \oplus y_2) \leq d(x_1, x_2) + d(y_1, y_2),
\[ii) \ d(x_1 \oplus y_1, x_2 \oplus y_2) \leq \max \{d(x_1, x_2), d(y_1, y_2)\}.
\]
For the construction of the pseudo-integral, a necessary notion is the notion of \( \sigma\odot\)-measure ([11]).

Let \( X \) be a non-empty set and let \( \Sigma \) be a \( \sigma\)-algebra of the subsets of the set \( X \). A set function \( \mu : \Sigma \to [a, b)_+ \) is a \( \sigma\odot\)-measure if
\[i) \ \mu(\emptyset) = 0,
\[ii) \ \mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \bigoplus_{i=1}^{\infty} \mu(A_i) = \lim_{n \to \infty} \bigoplus_{i=1}^{n} \mu(A_i),
\]
where \( \{A_i\}_{i \in \mathbb{N}} \) is a sequence of pairwise disjoint sets from \( \Sigma \). If \( \oplus \) is an idempotent operation, then the disjointness of sets and the condition \( i) \) can be omitted.

The pseudo-characteristic function of a set \( A \in \Sigma \) is given by
\[
\chi_A(x) = \begin{cases} 1, & \text{for } x \in A, \\ 0, & \text{for } x \not\in A. \end{cases}
\]
An elementary function is mapping \( e : X \to [a, b] \) that has the following representation:
\[
e = \bigoplus_{i=1}^{\infty} a_i \odot \chi_{A_i},
\]
where \( a_i \in [a, b], A_i \in \Sigma \) and \( \chi_A \) is the pseudo-characteristic function of a set \( A \).

Let \( \mu \) be a \( \sigma\odot\)-measure and \( \Sigma \) is a \( \sigma\)-algebra of the subsets of the non-empty set \( X \).

i) The pseudo-integral of an elementary function \( e \) with respect to \( \mu \) is defined by
\[
\int_X e \odot d\mu = \bigoplus_{i=1}^{\infty} a_i \odot \mu(A_i).
\]

ii) The pseudo-integral of a bounded measurable function \( f : X \to [a, b], (\text{see [11]}), \) is
\[
\int_X f \odot d\mu = \lim_{n \to \infty} \int_X \varphi_n \odot d\mu,
\]
where \( (\varphi_n)_{n \in \mathbb{N}} \) is a sequence of elementary functions chosen in such a manner that \( d(\varphi_n(x), f(x)) \to 0 \) uniformly while \( n \to \infty \) and \( d \) is the previously mentioned metric.

The previous limit is independent of the chosen sequence \( (\varphi_n)_{n \in \mathbb{N}} \) (see [11]). The pseudo-integral of the function \( f \) on some arbitrary subset \( A \) of \( X \) is given by
\[
\int_A f \odot d\mu = \int_X (f \odot \chi_{A}) \odot d\mu,
\]
where \( \chi_{A} \) is the pseudo-characteristic function. More on this construction can be found in [11].

A. Chebyshev integral inequality for the pseudo-integral

The generalization of the Chebyshev type inequality was proved in [1], [12]. In [1] authors proved two types of generalization of the Chebyshev integral inequality for the pseudo-integral for comonotone functions.

**Definition 1:** Functions \( f, g : X \to \mathbb{R} \) are said to be comonotone if for all \( x, y \in X, \)
\[
(f(x) - f(y))(g(x) - g(y)) \geq 0.
\]

**Theorem 2:** Let \( u, v : [0, 1] \to [a, b] \) be two measurable functions and let a generator \( g \) of the pseudo-addition \( \oplus \) and the pseudo-multiplication \( \odot \) be an increasing function. If \( g \circ u \) and \( g \circ v \) are comonotone, then the following inequality holds
\[
\left( \int_{[0,1]} u \odot d\mu \right) \odot \left( \int_{[0,1]} v \odot d\mu \right) \preceq \int_{[0,1]} (u \odot v) \odot d\mu.
\]

Any sup-measure generated as an essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to the generated pseudo-addition (see [10]).

For Lebesgue measure \( \nu \) on \( \mathbb{R} \) holds
\[
\mu(A) = \varepsilon \sup \{x \mid x \in A \} = \sup \{a \mid \nu \{x \mid x \in A, x > a\} > 0\}.
\]

**Theorem 3:** Let \( ([0, \infty], \sup, \odot) \) be a semiring with \( \odot \) generated by some increasing generator \( g \), i.e., \( x \odot y = g^{-1}(g(x)g(y)) \) for every \( x, y \in [a, b] \). Let \( \mu \) be a sup-measure on \( ([0,1], B([0,1]), \) where \( B([0,1]) \) is the Borel \( \sigma\)-algebra on \( [0,1], \) \( \mu(A) = \varepsilon \sup \{\psi(x) \mid x \in A\} \), and \( \psi : [0,1] \to [0, \infty] \) is a continuous density. If \( g \circ u \) and \( g \circ v \) are comonotone, then the following inequality holds
\[
\left( \sup_{[0,1]} u \odot d\mu \right) \odot \left( \sup_{[0,1]} v \odot d\mu \right) \preceq \sup_{[0,1]} (u \odot v) \odot d\mu.
\]

Two types generalization of Chebyshev’s inequality for the pseudo-integral of functions which are both increasing or both decreasing are proved in [12].
III. SET-VALUED PSEUDO-INTEGRAL

This section contains the generalization of the pseudo-integral done in the Aumann’s style ([2]). The first part is based on the integration of a set-valued function (see [3]), while the second part is focused on a more specific case, namely, the integration of interval-valued functions ([3], [5]).

Let \([a, b], \oplus, \odot\) be a semiring, \(X\) a nonempty set, \(\Sigma = \mathcal{P}(X)\) and \((X, \Sigma, \mu)\) measure space where \(\mu\) is a \(\sigma\)-\(\odot\)-measure. Let \(f : X \to [a, b]_+\) be a bounded and measurable function. Furthermore, let us suppose that the function \(f\) is integrable with respect to the measure \(\mu\) i.e., \[\int_X f \circ d\mu\] exists as a finite value in the sense of a semiring \(([a, b], \oplus, \odot)\) (see [5]). The family of all such functions \(f\) will be denoted with \(L^\oplus_{\odot}(\mu)\).

A. Pseudo-integration of set-valued functions

Let \(\mathcal{F}\) be the class of all closed subsets of \([a, b]_+\). A set-valued function \(F\) is a function from \(X\) to \(\mathcal{F}\). Further, only the measurable set-valued functions \(F : X \to \mathcal{F}\) shall be considered.

Remark 4: A set-valued function \(F : X \to \mathcal{F}\) is measurable if its graph is measurable, that is if \(\mathcal{F}\) is a function from \(X \to \mathcal{F}\) and \(F\) is measurable. \(\mathcal{F}\) is a measurable if its graph is measurable, that is if \(\mathcal{F}\) is a measurable.

Definition 5: The pseudo-integral of the set-valued function \(F\) on \(A \in \Sigma\) is
\[
\int_A F \oplus d\mu = \left\{ \int_A f \circ d\mu \mid f \in S(F) \right\},
\]  
(1)
where
\[
S(F) = \left\{ f \in L^\oplus_{\odot}(\mu) \mid f(x) \in F(x) \text{ on } X \mu - a.e. \right\}.
\]  
(2)

Specially, when a pseudo-integral coincides with the Lebesgue integral (see [11]), the integral (1) is the classical Aumann’s integral ([2]).

A set-valued function \(F : X \to \mathcal{F}\) is pseudo-integrable on \(A \in \Sigma\) if
\[
\int_A F \oplus d\mu \neq \emptyset.
\]

B. Pseudo-integration of interval-valued functions

This subsection is an overview of the results from [5].

Let \(\mathcal{I}\) be the class of all closed subintervals of \([a, b]_+\), i.e.,
\[
\mathcal{I} = \left\{ [u, v] \mid u \leq v \text{ and } [u, v] \subseteq [a, b]_+ \right\}.
\]

An interval-valued function is a function from \(X\) to \(\mathcal{I}\). The interval-valued function observed, with respect to the previously given measure space, is measurable.

Due to the specific range, interval-valued functions \(F\) are represented by means of the border functions \(l, r : X \to [a, b]_+\) in the following manner
\[
F(x) = [l(x), r(x)].
\]  
(3)

It is obvious that border functions are measurable and that they are selections of \(F\), i.e., \(l(x), r(x) \in F(x)\) for all \(x \in X\).

The pseudo-addition of two interval-valued functions and the pseudo-multiplication of an interval-valued function and an arbitrary real parameter from \([a, b]_+\) are, again, of the form (3) with corresponding border functions, see [5]. Let \(F_1\) and \(F_2\) be interval-valued functions represented by their border functions \(l_1, r_1, l_2\) and \(r_2\) as \(F_1(x) = [l_1(x), r_1(x)]\) and \(F_2(x) = [l_2(x), r_2(x)]\). Then, based on the representation (3) and the properties of operations \(\oplus\) and \(\odot\) from three basic classes of semirings, holds
\[
(F_1 \oplus F_2)(x) = F_1(x) \oplus F_2(x) = [l_1(x) \oplus l_2(x), r_1(x) \oplus r_2(x)]
\]
and for some \(\alpha \in [a, b]_+\) holds
\[
(\alpha \odot F_1)(x) = \alpha \odot F_1(x) = [\alpha \odot l_1(x), \alpha \odot r_1(x)].
\]

C. Pseudo-integrability of set-valued functions

Pseudo-integrability of set-valued functions is closely connected to the following property of set-valued functions (see [3], [5]).

Definition 6: A set-valued function \(F\) is pseudo-integrably bounded if there is a function \(h \in L^\oplus_{\odot}(\mu)\) such that:

i) \(\bigoplus \alpha \leq h(x)\), for the idempotent pseudo-addition,

ii) \(\sup \alpha \leq h(x)\), for the pseudo-addition given by an increasing generator \(g\),

iii) \(\inf \alpha \leq h(x)\), for the pseudo-addition given by a decreasing generator \(g\).

Now, a sufficient condition for the pseudo-integrability of a set-valued function \(F\) is given by the following proposition.

Proposition 7: If \(F\) is a pseudo-integrably bounded set-valued function, then \(F\) is pseudo-integrable.

In order to formulate the Chebyshev type inequality for the set-valued pseudo-integral, it is necessary to introduce the relation "less or equal" applied on sets from \(\mathcal{F}\) :

- if \(C, D \in \mathcal{F}\) and \(C \subseteq D\) then for all \(x \in C\) there exists \(y \in D\) such that \(x \preceq y\) and for all \(y \in D\) there exists \(x \in C\) such that \(x \preceq y\).

The following theorem was proved in [5].

Theorem 8: Let \(F\) be a pseudo-integrably bounded interval-valued function with border functions \(l\) and \(r\). Then,
\[
\int_X F \circ d\mu = \left[ \int_X l \circ d\mu, \int_X r \circ d\mu \right].
\]

IV. CHEBYSHEV INEQUALITY FOR THE SET-VALUED PSEUDO-INTEGRAL

This section contains the main results of this paper. In the first part of this section the Chebyshev type inequality for monotone pseudo-integrable set-valued functions will be proved. The second part of this section will give the Chebyshev type inequality for pseudo-integrably bounded interval-valued functions.
If $A$ and $B$ are non-empty sets, the following notation will be used
\[ A \odot B = \{ c \odot d \mid c \in A, d \in B \}. \tag{4} \]

**Definition 9:** A set-valued function $F : X \rightarrow \mathcal{F} \setminus \emptyset$ is monotone if, for all $x, y \in X$, $u \in F(x)$ and $v \in F(y)$,
\[ (x - y) (g \circ u - g \circ v) \geq 0, \]
where $g$ is a generator of the pseudo-operations for the $g$-semiring or $g$ is a generator of $\odot$ if the semiring $([a, b], \max, \odot)$, where $\odot$ is generated is considered.

**Theorem 10:** Let $F_1, F_2 : [0, 1] \rightarrow \mathcal{F} \setminus \emptyset$ be monotone pseudo-integrable set-valued functions such that for all $f_1 \in S(F_1)$ and $f_2 \in S(F_2)$ holds $f_1 \odot f_2 \in L^1_{\odot}(\mu)$. If a generator $g$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is an increasing function, then holds:
\[ \left(\int_{[0,1]} \oplus \left(\begin{array}{c} F_1 \odot d\mu \\ F_2 \odot d\mu \end{array}\right)\right) \preceq_S \int_{[0,1]} (F_1 \odot F_2) \odot d\mu. \tag{5} \]

**Proof.** Since $F_1, F_2$ are pseudo-integrable set-valued functions on $[0, 1]$, it is $\int_{[0,1]} F_i \odot d\mu \neq \emptyset$, $i = 1, 2$.

For $u \in \left(\int_{[0,1]} F_1 \odot d\mu \right) \oplus \left(\int_{[0,1]} F_2 \odot d\mu \right)$ (by definition of the pseudo-integral of the set-valued function and (4)), there are functions $f_1 \in S(F_1)$ and $f_2 \in S(F_2)$ such that $u = \left(\int_{[0,1]} f_1 \odot d\mu \right) \oplus \left(\int_{[0,1]} f_2 \odot d\mu \right)$. Since $F_1$ and $F_2$ are monotone set-valued functions, by Definition 9, for all $x, y \in [0, 1], f_1(x) \in F_1(x)$ and $f_2(y) \in F_2(y), i = 1, 2$ holds
\[ (x - y) (g \circ f_1(x) - g \circ f_1(y)) \geq 0, i = 1, 2, \]
i.e.
\[ (g \circ f_1(x) - g \circ f_1(y)) (g \circ f_2(x) - g \circ f_2(y)) \geq 0. \]
Therefore $g \circ f_1$ and $g \circ f_2$ are comonotone functions.

From the Theorem 2 follows
\[ \left(\int_{[0,1]} f_1 \odot d\mu \right) \oplus \left(\int_{[0,1]} f_2 \odot d\mu \right) \leq \left(\int_{[0,1]} (f_1 \circ f_2) \odot d\mu \right). \tag{6} \]

From $f_1 \odot f_2 \in L^1_{\odot}(\mu)$ and (4) follows $f_1 \circ f_2 \in S(F_1 \odot F_2)$ and
\[ v = \left(\int_{[0,1]} (f_1 \circ f_2) \odot d\mu \right) \geq \left(\int_{[0,1]} (F_1 \odot F_2) \odot d\mu \right), \]
and $u \preceq v$.

For $v \in \left(\int_{[0,1]} (F_1 \circ F_2) \odot d\mu \right)$ exists $f_1 \circ f_2 \in S(F_1 \odot F_2) \in \mathcal{F} \setminus \emptyset$ such that $v = \left(\int_{[0,1]} (f_1 \circ f_2) \odot d\mu \right)$. Since $g \circ f_1$ and $g \circ f_2$ are comonotone functions from Theorem 2 inequality (6) follows. By the inequality (6) and $f_1 \odot f_2 \in L^1_{\odot}(\mu)$ hold $f_1 \in L^1_{\odot}(\mu)$, $f_2(x) \in F_1(x)$ $\mu$-a.e. on $[0, 1]$ and $f_2 \in L^1_{\odot}(\mu)$, $f_2(x) \in F_2(x)$ $\mu$-a.e. on $[0, 1]$, i.e. $f_1 \in S(F_1)$ and $f_2 \in S(F_2)$.

Now, from the Theorem 2 it follows
\[ u = \left(\int_{[0,1]} f_1 \odot d\mu \right) \oplus \left(\int_{[0,1]} f_2 \odot d\mu \right) \leq v. \]

The following theorem can be proved in an analogous manner.

**Theorem 11:** Let $F_1, F_2 : [0, 1] \rightarrow \mathcal{F} \setminus \emptyset$ be monotone pseudo-integrable set-valued functions such that for all $f_1 \in S(F_1)$ and $f_2 \in S(F_2)$ holds $f_1 \odot f_2 \in L^1_{\odot}(\mu)$. If $\odot$ is represented by an increasing generator $g$ and $m$ be the same as in the Theorem 3, then holds:
\[ \left(\int_{[0,1]} \oplus \left(\begin{array}{c} \sup \limits_{[0,1]} F_1 \odot d\mu \\ \sup \limits_{[0,1]} F_2 \odot d\mu \end{array}\right)\right) \preceq_S \int_{[0,1]} (F_1 \odot F_2) \odot d\mu. \tag{7} \]

The example below will show that the assertion of the Theorem 10 is not valid if for $f_1 \in S(F_1)$ and $f_2 \in S(F_2)$ does not hold $f_1 \odot f_2 \in L^1_{\odot}(\mu)$.

**Example 12:** For $[a, b] = [0, \infty]$ and $g(x) = \sqrt{x}$ hold
\[ x \oplus y = \sqrt{x^2 + y^2}, \quad x \circ y = x \cdot y \quad \text{and} \quad 0 = 0. \]

If $F_i(x) = \left\{ 0, \frac{1}{1-x} \right\}, i = 1, 2,$
then
\[ (F_1 \circ F_2)(x) = \left\{ 0, \frac{1}{(1-x)^2} \right\}. \]

From
\[ \int_{[0,1]} \frac{1}{1-x} \odot d\mu = 4 \quad \text{and} \quad \int_{[0,1]} \frac{1}{(1-x)^2} \odot d\mu = \infty, \]
holds
\[ \int_{[0,1]} F_i \odot d\mu = \{ 0, 4 \}, \quad i = 1, 2, \]
ie.
\[ \left(\int_{[0,1]} F_1 \odot d\mu \right) \oplus \left(\int_{[0,1]} F_2 \odot d\mu \right) = \{ 0, 16 \} \]
and
\[ \int_{[0,1]} (F_1 \circ F_2) \odot d\mu = \{ 0 \}. \]
It is obvious that the assertion of the Theorem 10 does not hold.

**Theorem 13:** Let \( F_1, F_2 : [0, 1] \to \mathcal{F}\setminus\{\emptyset\} \) be monotone, pseudo-integrable bounded set-valued functions and \( F_1 \oplus F_2 \) is a pseudo-integrably bounded set-valued function. If a generator \( g \) of the pseudo-addition \( \oplus \) and the pseudo-multiplication \( \odot \) is an increasing function, then inequality (5) holds.

**Proof.** \( F_1 \oplus F_2 \) is a pseudo-integrably bounded set-valued function and \( h \) is the function from the Definition 6.

Let us observe the arbitrary function \( f_1 \odot f_2 : [0, 1] \to [a, b]_+ \) with the property \( (f_1 \odot f_2)(x) \in (F_1 \odot F_2)(x) \). \( f_1 \odot f_2 \) is a measurable function. Since \( h \) is from \( L^1_\mu(\mu) \), it is bounded and ensures the boundness of the function \( f_1 \odot f_2 \), i.e., for all \( x \in [0, 1] \) and some \( M \in (a, b)_+ \) holds

\[
(f_1 \odot f_2)(x) \leq \sup_{\alpha \in F_1(x) \odot F_2(x)} \alpha \leq h(x) \leq M \cdot b.
\]

Now, the properties of the pseudo-integral (see [11]) will ensure

\[
\int_{[0,1]} (f_1 \odot f_2) \odot d\mu \leq \int_{[0,1]} h \odot d\mu.
\]

Since \( h \) is an integrable function, function \( f_1 \odot f_2 \in L^1_\mu(\mu) \) and by the Theorem 11, the inequality (5) holds. \( \square \)

The following theorem can be shown.

**Theorem 14:** Let \( F_1, F_2 : [0, 1] \to \mathcal{F}\setminus\{\emptyset\} \) be monotone, pseudo-integrably bounded set-valued functions and \( F_1 \odot F_2 \) is a pseudo-integrably bounded set-valued function. If \( \odot \) is represented by an increasing generator \( g \) and \( m \) be the same as in the Theorem 3, then the inequality (7) holds.

In the example below monotone, pseudo-integrable set-valued functions \( F_1, F_2 : [0, 1] \to \mathcal{F}\setminus\{\emptyset\} \) such that for all \( f_1 \in S(F_1) \) and \( f_2 \in S(F_2) \) holds \( f_1 \odot f_2 \in L^1_\mu(\mu) \) but \( F_1 \odot F_2 \) is not a pseudo-integrably bounded set-valued function are considered.

**Example 15:** For \([a, b] = [-\infty, \infty]\) and \( g(x) = e^x \)

\[
x \oplus y = \ln(e^x + e^y) \quad \text{and} \quad x \odot y = x + y.
\]

If

\[
F_1(x) = \left\{ -\frac{1}{3} \ln (1 - x) \right\}, \quad F_2(x) = \left\{ -\frac{1}{6} \ln (1 - x) \right\},
\]

then

\[
(F_1 \odot F_2)(x) = \left\{ -\frac{1}{2} \ln (1 - x) \right\}.
\]

From

\[
\int_{[0,1]} \left( -\frac{1}{3} \ln (1 - x) \right) \odot d\mu = \ln \frac{3}{2}.
\]

(\( \square \))

and

\[
\int_{[0,1]} \left( -\frac{1}{2} \ln (1 - x) \right) \odot d\mu = \ln \frac{6}{5},
\]

(\( \square \))

and (7) obviously holds.

Next theorem follows from the Theorem 2, the Theorem 3 and the Theorem 8.

**Theorem 16:** Let \( F_1, F_2 : [0, 1] \to \mathcal{F}\setminus\{\emptyset\} \) be pseudo-integrably bounded interval-valued functions represented by their border functions \( l_1, r_1, l_2 \) and \( r_2 \). Let \( F_1 \odot F_2 \) be a pseudo-integrably bounded interval-valued function and a generator \( g \) of the pseudo-addition \( \oplus \) and the pseudo-multiplication \( \odot \) is an increasing function. If \( g \circ l_1 \) and \( g \circ l_2 \) are comonotone and \( g \circ r_1 \) and \( g \circ r_2 \) are comonotone then holds:

\[
\left[ \int l_1 \odot d\mu, \int r_1 \odot d\mu \right] \odot \left[ \int l_2 \odot d\mu, \int r_2 \odot d\mu \right] \leq S \left[ \int (l_1 \circ l_2) \odot d\mu, \int (r_1 \circ r_2) \odot d\mu \right].
\]

(8)

**Theorem 17:** Let \([a, b], \max, \odot \) be a semiring where \( \odot \) is represented by an increasing generator \( g \) and \( m \) be the same as in the Theorem 3. Let \( F_1, F_2 : [0, 1] \to \mathcal{F}\setminus\{\emptyset\} \) be pseudo-integrably bounded interval-valued functions represented by their border functions \( l_1, r_1, l_2 \) and \( r_2 \). If \( F_1 \odot F_2 \) is a pseudo-integrably bounded interval-valued function and \( g \circ l_1 \) and \( g \circ l_2 \) are comonotone and \( g \circ r_1 \) and \( g \circ r_2 \) are comonotone then holds:

\[
\left[ \sup \left[ \int l_1 \odot d\mu, \int r_1 \odot d\mu \right], \sup \left[ \int l_2 \odot d\mu, \int r_2 \odot d\mu \right] \right] \leq S \left[ \sup \left[ \int (l_1 \circ l_2) \odot d\mu, \int (r_1 \circ r_2) \odot d\mu \right] \right].
\]
Chebyshev type inequality for functions \( f_1, f_2 : [0, 1] \rightarrow [a, b] \) when \( f_1 \) and \( f_2 \) are both increasing or decreasing for two cases of semirings is shown in [12]. Chebyshev type inequality for interval-valued functions \( F_1 \) and \( F_2 \) is true if their border functions \( l_1 \) and \( r_1 \) \((l_2 \text{ and } r_2)\) are with the same monotonicity (Example 18) or not (Example 19).

Example 18: For \([a, b] = [0, \infty]\) and \( g(x) = \sqrt{x}\),
\[
x \oplus y = \sqrt{x^3 + y^3} \quad \text{and} \quad x \odot y = x \cdot y.
\]
For \( F_1(x) = [x^6, x], F_2(x) = [8x^3, (2 + x)^3] \), holds
\[
(\bigcirc F_1 \odot F_2)(x) = \left[8x^9, x(2 + x)^3\right].
\]
From the Theorem 8 follows
\[
\int_{[0,1]} F_1 \odot d\mu = \left[\int_{[0,1]} x^6 \odot d\mu, \int_{[0,1]} x \odot d\mu\right] = \left[\frac{1}{27} \frac{27}{64}\right],
\]
\[
\int_{[0,1]} F_2 \odot d\mu = \left[\int_{[0,1]} 8x^3 \odot d\mu, \int_{[0,1]} (2 + x)^3 \odot d\mu\right] = \left[1, \frac{125}{8}\right],
\]
and
\[
\int_{[0,1]} (F_1 \odot F_2) \odot d\mu = \left[\int_{[0,1]} 8x^9 \odot d\mu, \int_{[0,1]} (2 + x)^3 \odot d\mu\right] = \left[\frac{1}{27} \frac{125}{64}\right],
\]
and the inequality (8) holds.

Example 19: Let \([a, b] = [0, 1]\), \( g(x) = \frac{x}{1 - x} \). Then
\[
g^{-1}(x) = \frac{x}{1 + x},
\]
x \(\oplus\) y = \( \frac{x + y - 2xy}{1 - xy} \) and x \(\odot\) y = \( \frac{xy}{2xy - y - x + 1} \).
If \( F_1(x) = \left[\frac{x}{1 + x}, \frac{4 - x}{5 - x}\right], F_2(x) = \left[\frac{x^2}{1 + x^2}, \frac{3 - x}{4 - x}\right], \)
(border functions \( l_1 \) and \( l_2 \) are increasing functions, \( r_1 \) and \( r_2 \) are decreasing functions) then
\[
(\bigcirc F_1 \odot F_2)(x) = \left[\frac{x^3}{1 + x^3}, \frac{x^2 - 7x + 12}{x^2 - 7x + 13}\right].
\]
From the Theorem 8 follows
\[
\int_{[0,1]} F_1 \odot d\mu = \left[\frac{1}{3} \frac{7}{9}\right], \quad \int_{[0,1]} F_2 \odot d\mu = \left[\frac{1}{4} \frac{5}{7}\right]
\]
and
\[
\int_{[0,1]} (F_1 \odot F_2) \odot d\mu = \left[\frac{1}{5} \frac{53}{59}\right].
\]
and the inequality (8) holds.

V. CONCLUSION

This paper proved the Chebyshev type integral inequality for the pseudo-integral of the set-valued functions and the interval valued functions. The future work can be directed to the research of the possible applications of the obtained results.

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