On restricted distributivity of nullnorm with respect to t-conorm and corresponding utility function

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Abstract—The problem of the restricted distributivity, i.e., distributivity on the relaxed domain, plays an important role in many different fields such as the utility theory and the integration theory. This paper considers restricted distributivity of a continuous nullnorm with respect to a continuous t-conorm and application those operators in the utility theory.

Keywords: absorbing element, nullnorms, restricted distributivity, utility function.

I. INTRODUCTION

While working with different types of operators defined on an interval of reals that are essential for decision making theory, fuzzy set theory, theory of optimization, integration theory, etc., as one of the crucial issues appears the problem of distributivity. Investigation of this problem has roots in [1] and, in recent years, it has been directed towards aggregation operators [2], fuzzy implications [22], [23], uninorms and nullnorms [4], [9], [17], [21], distributivity inequalities ([5], [6]). Of the special interest for this paper is the restricted distributivity, also known as the conditional distributivity, i.e., distributivity on the relaxed domain. Different aspects of this particular problem are the main topics of the number of research papers, such as [8], [14], [15], [13], [24], [25]. While [8] focused on conditional distributivity of a t-norm over a t-conorm and utility function, the problem of integration, namely of \((S,T)\)-integral, \((S,U)\)-integral and general fuzzy integral, is investigated in [14], [15], [13], [25].

The aim of this paper is to extend the previous research towards operators with annihilator, i.e., with absorbing element, that can be applicable in the utility theory for modeling behavior of the decision maker. Therefore, the focus is now on the distributivity equation

\[
F(x,G(y,z)) = G(F(x,y),F(x,z)), \quad x,y,z \in [0,1], \quad G(y,z) < 1
\]

where \(F\) a continuous nullnorm.

The paper is organized as follows. Section 2 contains preliminary notions concerning nullnorms, restricted distributivity and hybrid utility function. Results on restricted distributivity for continuous nullnorms with respect to continuous t-conorm are given in the third section. Section 4 is devoted to application obtained results in the utility theory. Some concluding remarks are given in the fifth section.

II. PRELIMINARIES

A short overview of the basic notions that are essential for this paper is given in this section ([3], [10], [14], [16], [17], [26], [8]).

A. Nullnorms

First type of operator necessary for the presented research is a nullnorm such that

\[
F(x,k) = k \text{ for } x \in [0,1],
\]

\[
F(x,0) = x \text{ for } x \leq k \quad \text{and} \quad F(x,1) = x \text{ for } x \geq k.
\]

Again, triangular norms and triangular conorms can be obtained as the special cases of the previous definition. Now, for \(k = 0\) operator \(F\) is a t-norm and for \(k = 1\) it is a t-conorm. Representation of nullnorms for \(k \in (0,1)\), i.e., for a nullnorm that is not a t-norm, nor a t-conorm is given by the following theorem from [16].

Theorem 2: ([16]) Let \(F\) be a nullnorm such that \(k \in (0,1)\). Then

\[
F(x,y) = \begin{cases} \frac{k S \left( \frac{x}{k}, \frac{y}{k} \right)}{k + (1-k) T \left( \frac{x}{1-k}, \frac{y}{1-k} \right)} & \text{if } (x,y) \in [0,k]^2, \\ k & \text{otherwise}, \end{cases}
\]

where \(T\) is a t-norm, and \(S\) is a t-conorm.

\[(1)\]
B. Restricted distributivity

Since the problem of non restricted distributivity of a t-norm (or a uninorm) over a t-conorm gives us only the trivial solution, that is, a t-conorm in question has to be \( S_m = \max \), it was necessary to relax the domain of distributivity in the following manner ([14]).

Definition 3: A t-norm \( T \) is restrictedly distributive (RD) over a t-conorm \( S \) if for all \( x, y, z \in [0,1] \) the following holds

\[
T(x,S(y,z)) = S(T(x,y),T(x,z))
\]

whenever \( S(y,z) < 1 \).

This type of distributivity is also known as the conditional distributivity ([14]) and, although the domain is only weakly relaxed, the class of pairs of operators that fulfill (RD) is much wider. The following theorem illustrates this for a continuous t-norm and a continuous t-conorm (see [14]).

Theorem 4: [14] A continuous t-norm \( T \) and a continuous t-conorm \( S \) satisfy (RD), if and only if exactly one of the following cases is fulfilled:

(i) \( S = S_m = \max \)

(ii) there is a strict t-norm \( T^* \) and a nilpotent t-conorm \( S^* \) such that additive generator \( s \) of \( S^* \) satisfying \( s(1) = 1 \) is also a multiplicative generator of \( T^* \), and there is an \( a \in [0,1] \) such that for some continuous t-norm \( T^{**} \) the following holds:

\[
S(x,y) = \begin{cases} 
  a + (1-a)S^* \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [a,1]^2, \\
  \max(x,y) & \text{otherwise,}
\end{cases}
\]

and

\[
T(x,y) = \begin{cases} 
  aT^{**} \left( \frac{y-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [0,a]^2, \\
  a + (1-a)T^* \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [a,1]^2, \\
  \min(x,y) & \text{otherwise.}
\end{cases}
\]

Remark 5: (i) Due to isomorphisms between strict t-norms and product t-norm \( T_p(x,y) = xy \) and nilpotent t-conorms and Lukasiewicz t-conorm \( S_L(x,y) = \min(x+y,1) \), the previous result is often reduced to the pair

\[
S(x,y) = \begin{cases} 
  a + (1-a)S_L \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [a,1]^2, \\
  \max(x,y) & \text{otherwise,}
\end{cases}
\]

and

\[
T(x,y) = \begin{cases} 
  aT^{**} \left( \frac{y-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [0,a]^2, \\
  a + (1-a)T_p \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [a,1]^2, \\
  \min(x,y) & \text{otherwise,}
\end{cases}
\]

given by Figure 1 (see [14]).

(ii) This result has had a drastic impact on the utility theory and notion of mixtures. It has allowed the construction of the hybrid mixture that is possibilistic under a certain threshold and probabilistic above it (see [8]).

(iii) Further generalizations of Theorem 4 can be found in [13], where \( F \) is a left-continuous uninorm, and in [25], where \( F \) is a pseudo-multiplication.

C. Hybrid mixtures

Let \((S,T)\) be pair of continuous t-conorm and t-norm satisfying (RD) and \( \Phi \), set of ordered pairs defined in the following way \( \Phi = \{ (\alpha,\beta) \mid \alpha, \beta \in [0,1], S(\alpha,\beta) = 1 \} \). Now, set \( \Phi_{S,a} \subset \Phi \) is given by the following

\[
\Phi_{S,a} = \{ (\alpha,\beta) \mid \alpha, \beta \in (a,1), \alpha + \beta = 1 + a \mbox{ or } \min(\alpha,\beta) \leq a, \max(\alpha,\beta) = 1 \}.
\]

The following definition from [8] of is an extension of the definition of the mixture sets investigated in [7] and [12].

Definition 6: ([8]) A hybrid mixture set is a quadruple (G,M,T,S) such that \((S,T)\) is a pair of continuous t-conorm and t-norm that satisfy (RD), \( G \) is a set, and \( M : G^2 \times \Phi_{S,a} \rightarrow G \) is a function (hybrid mixture operation) given by

\[
M(x,y;\alpha,\beta) = S(T(\alpha,x),T(\beta,y)).
\]

Also, the optimistic hybrid utility function by means of hybrid mixtures is given by the following (see [8]):

\[
U(u_1,u_2;\mu_1,\mu_2) = S(T(u_1,\mu_1),T(u_2,\mu_2))
\]

where \( u_1, u_2 \) are two utilities with values in \([0,1]\) and \( \mu_1, \mu_2 \) are two degrees of plausibility from \( \Phi_{S,a} \).

III. Restricted Distributivity for Continuous Nullnorms

This section contains a characterization of all pairs \((F,G)\) satisfying (RD) where \( F \) is a continuous nullnorm with non trivial absorbing element and \( G \) is a continuous t-conorm.

Theorem 7: A continuous nullnorm \( F \) with absorbing element \( k \in (0,1) \) and a continuous t-conorm \( S \) satisfy (RD) if and only if exactly one of the following cases is fulfilled:

(i) \( S = S_m = \max \)

(ii) there is an \( a \in [1,1] \) such that \( S \) is of the form (4) and \( F \) is given by:

\[
F(x,y) = \begin{cases} 
  kS_1 \left( \frac{x}{a}, \frac{y}{a} \right) & \text{if } (x,y) \in [0,k]^2, \\
  k + (a-k)T_1 \left( \frac{x}{a-k}, \frac{y}{a-k} \right) & \text{if } (x,y) \in [k,a]^2, \\
  a + (1-a)T_F \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x,y) \in [a,1]^2, \\
  \min(x,y) & \text{if } k \leq \min(x,y) \leq a \leq \max(x,y), \\
  k & \text{otherwise,}
\end{cases}
\]

Figure 1. Restricted distributivity: t-norm and t-conorm.
Figure 2. Restricted distributivity: nullnorm and t-conorm.

where $S_1$ is an arbitrary continuous t-conorm and $T_1$ is an arbitrary continuous t-norm (see Figure 2).

**Proof:** First, let us suppose that $F$ and $S$ are a continuous nullnorm and a continuous t-conorm that satisfy (RD):

- For $x \leq k$ holds $x = F(x, S(0, 0)) = S(F(x, 0), F(x, 0)) = S(x, x)$, i.e., all $x \in [0, k]$ are idempotent elements for $S$.

- Let $x \geq k$. If $a \in [k, 1)$ is an idempotent element for $S$, then for all $x \in [k, 1)$ holds

\[
F(x, a) = F(x, S(a, a)) = S(F(x, a), F(x, a)),
\]

that is, $F(x, a)$ is also an idempotent element for $S$. Due to continuity of $F$, range of function $F(\cdot, a)$ for input values from $[k, 1)$ is $[k, a]$, which insures that all elements from $[k, a]$ are idempotent for $S$.

Hence, all elements from $[0, 1]$ are idempotent for t-conorm $S$. Therefore, $S = S_M = \max$, or there is the largest nontrivial idempotent element $a \in [k, 1)$ of $S$ such $S$ is of the form (4). Equality (7) now follows from Theorem 4 and Theorem 2.

Conversely, if $S$ is a t-conorm of the form (4) and $F$ a nullnorm of the form (7), it can be easily shown that condition (RD) holds. For input values from $[a, 1]$ the problem is narrowed down to the pair $(T_P, S_L)$ which satisfies (RD), and in all other cases it follows from fact that $S = \max$ while $F$ is increasing.

**Remark 8:** If for the previous problem is observed nonrestricted distributivity, only the trivial solution is obtained, i.e., $S = S_M$ (see [2]).

**IV. RESTRICTED DISTRIBUTIVITY AND UTILITY FUNCTION**

In this section it is used a continuous nullnorm $F$ from the previous section instead of continuous t-norm $T$ in definition hybrid utility function (6) in order to obtain a new extension. Now

\[
U_F(u_1, u_2; \mu_1, \mu_2) = S(F(u_1, \mu_1), F(u_2, \mu_2))
\]

where $u_1, u_2$ are two utilities with values in $[0, 1]$ and $\mu_1, \mu_2$ are two degrees of plausibility from $\Phi_{S,a}$.

Examination of the behavioral characteristics of this new utility function follows.

**Case I** Let $\mu_1 > a, \mu_2 > a$, i.e., $\mu_1 + \mu_2 = 1 + a$. Now the following subcases have to be considered.

1) Let $u_1, u_2 > a$. Then

\[
U_F(u_1, u_2; \mu_1, \mu_2) = \frac{u_1(\mu_1 - a) + u_2(1 - \mu_1)}{1 - a}.
\]

2) Let $u_1 \leq a, u_2 > a$. Then

\[
U_F(u_1, u_2; \mu_1, \mu_2) = a + \frac{(u_2 - a)(\mu_2 - a)}{1 - a}.
\]

3) Let $u_1 > a, u_2 \leq a$. Then

\[
U_F(u_1, u_2; \mu_1, \mu_2) = a + \frac{(u_1 - a)(\mu_1 - a)}{1 - a}.
\]

4) Let $u_1, u_2 \leq a$. Then:

\[\begin{align*}
&\text{a)} \text{ if } u_1, u_2 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = k; \\
&\text{b)} \text{ if } k < u_1 \leq a, u_2 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = \max(u_1, k) = u_1; \\
&\text{c)} \text{ if } k < u_2 \leq a, u_1 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = \max(u_2, k) = u_2; \\
&\text{d)} \text{ if } k < u_1 \leq a, k < u_2 \leq a, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = \max(u_1, u_2).
\end{align*}\]

For the first three subcases situation is the same as for hybrid utility function $U$ (see [8]). However, the subcase 4 is of the special interest for the further investigation because it shows that when both utilities $u_1$ and $u_2$ are less than $k$, value of utility function $U$ is $k$, i.e., cannot be less than $k$. This property opens a door for discussion on utility functions with previously imposed threshold.

**Case II** Let $\mu_1 \leq a, \mu_2 = 1$ (the corresponding analysis of the case $\mu_2 \leq a, \mu_1 = 1$ can be done in an analogous way). Now the following subcases are being considered.

1) Let $\mu_1 \leq k$. Then the following holds.

\[\begin{align*}
&\text{a)} \text{ if } u_1, u_2 > a, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, u_2) = u_2; \\
&\text{b)} \text{ if } u_1, u_2 \leq a, \text{ then:} \\
&\quad \text{i)} \text{ if } u_2 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = k; \\
&\quad \text{ii)} \text{ if } u_2 > k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = u_2.
\end{align*}\]

2) Let $\mu_1 > k$. Then the following holds.

\[\begin{align*}
&\text{a)} \text{ if } u_1, u_2 > a, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = u_2; \\
&\text{b)} \text{ if } u_1, u_2 \leq a, \text{ then:} \\
&\quad \text{i)} \text{ if } u_1, u_2 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = k; \\
&\quad \text{ii)} \text{ if } u_1 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = \max(u_1, k) = u_1; \\
&\quad \text{iii)} \text{ if } u_2 \leq k, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = F(u_1, \mu_1); \\
&\quad \text{iv)} \text{ if } k < u_1 \leq a, k < u_2 \leq a, \text{ then } U_F(u_1, u_2; \mu_1, \mu_2) = \max(F(u_1, \mu_1), u_2).
\end{align*}\]

From the subcase 1) when the degree of plausibility $\mu_1 \leq k$, it can be seen that the value of the utility function $U_F$ is
either $u_2$ or $k$. When utility $u_2 \leq k$ then the value of the utility function $U_F$ is $k$, otherwise is $u_2$. From the subcase 2) when the degree of plausibility $k < \mu_1 \leq a$, follows that the value of the utility function $U_F$ cannot be less than $k$. Therefore, again the previously imposed threshold appears.

V. CONCLUSION

The focus of this paper is on distributivity equations involving a continuous nullnorm and a continuous t-conorm. It is shown that a bigger variety of solutions is obtained if there are required on the restricted domain. Also, if using a continuous nullnorm $F$ with absorbing element $k$, instead of a continuous t-norm $T$, in the definition of hybrid utility function $U$, its value cannot be less then $k$, i.e., a threshold that is imposed by the absorbing element of a continuous nullnorm is present. This property can be very useful for modeling behavior of some decision makers, therefore the further investigations will go to this direction.

Acknowledgment. This paper has been supported by the Ministry of Science and Technological Development of Republic of Serbia and by the Provincial Secretariat for Science and Technological Development of Vojvodina.

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