Constructions and Classifications on Involutive Residuated Semigroups

Sándor Jenei
Institute of Mathematics and Informatics, University of Pécs, Pécs, Hungary, jenei@ttk.pte.hu
Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria, sandor.jenei@jku.at

Abstract—Classification results of residuated lattices are presented in this paper along with some construction methods. Recent results are the classification of some classes of finite, involutive FL$_e$-algebras, of strongly involutive uninorm algebras, and of absorbent-continuous, sharp FL$_e$-algebras on weakly real chains. A new ordinal sum construction in introduced which constructs sharp, involutive FL$_e$-algebras.

Index Terms—Residuated lattice, involutive FL$_e$-algebra, generalized ordinal sum, twin-rotation, classification

I. INTRODUCTION

Some recent results concerning construction, structural description, and classification of classes of involutive FL$_e$-monoids is presented in this paper.

Residuated lattices have been introduced in the 30s of the last century by Ward and Dilworth [24] to investigate ideal theory of commutative rings with unit. Examples of residuated lattices include Boolean algebras, Heyting algebras [18], MV-algebras [3], BL-algebras, [8] and lattice-ordered groups; a variety of other algebraic structures can be rendered as residuated lattices. The topic did not become a leading trend on its own right back then. Nowadays the investigation of residuated lattices (roughly, residuated monoids on lattices) has got a new impetus and has been staying in the focus of strong international attention. Beyond the algebraic interest, the reason is that residuated lattices turned out to be algebraic counterparts of substructural logics [23], [22]. Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic and their non-commutative versions. These logics had different motivations and methodology. The theory of substructural logics has put all these logics, along with many others, under the same motivational and methodological umbrella. Residuated lattices themselves have been the key component in this remarkable unification. An extensive monograph about residuated lattices and substructural logics appeared in 2007 [7]. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science. FL$_e$-algebras are commutative residuated lattices with an additional constant. For FL$_e$-algebras, those with an involutive negation are of special interest. Involutive FL$_e$-algebras have very interesting symmetry properties [11], [13], [10], [19] and, as a consequence, for involutive FL$_e$-algebras we have beautiful geometric constructions which are lacking for general FL$_e$-algebras [11], [16], [20]. Furthermore, not only involutive FL$_e$-algebras have very interesting symmetry properties, but some of their logical calculi have important symmetry properties too: Both sides of a sequent may contain more than one formula, while (hyper)sequent calculi for their non-involutive counterparts admit at most one formula to the right.

As for the classification problem of residuated lattices, as one naturally expects, it is possible only by imposing additional postulates. A first precursor is due to Hölder who proved in [9] that every cancellative, Archimedean, naturally and totally ordered semigroup can be embedded into the additive semigroup of the real numbers. Aczél used tools of analysis to investigate continuous semigroup operations over intervals of real numbers and also found in [1, page 256] the cancellative property to be sufficient and necessary for the existence of an order-isomorphism to a subsemigroup of the additive semigroup of the real numbers [1, page 268]. Clifford showed in [4] that every Archimedean, naturally and totally ordered semigroup in which the cancellation law does not hold can be embedded into either the real numbers in the interval [0,1] with the usual ordering and $ab = \max(a + b, 1)$ or the real numbers in the interval [0,1] and the symbol $\infty$ with the usual ordering and $ab = a + b$ if $a + b \leq 1$ and $ab = \infty$ if $a + b > 1$. For a summary of the Hölder and Clifford theorems, see [5, Theorem 2 in Section 2 of Chapter XI]. Clifford also introduced the ordinal sum construction for a family of totally ordered semigroups in [4] and proved that every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups. Mostert and Shields gave a complete description of topological semigroups over compact manifolds with connected, regular boundary in [21] by using a subclass of compact connected Lie groups and via classifying semigroups on arcs such that one endpoint functions as an identity for the semigroup, and the other functions as a zero. They classified such semigroups as ordinal sums of three basic multiplications which an arc may possess. The word ‘topological’ refers to the continuity of the semigroup operation with respect to the topology. In the next related classification result, the topologically connected property of the underlying chain was dropped whereas the continuity condition was somewhat strengthened: Under the assumption

\[ \text{Isotonicity of the semigroup operation is not assumed.} \]

\[ \text{He called is reducible.} \]
of divisibility, residuated chains were classified as ordinal sums of linearly ordered Wajsberg hoops in [2]. Postulating the divisibility condition proved to be sufficient for the classification of semi-linear residuated monoids over arbitrary lattices, see [17], where the authors introduced the notion of poset sum of hoops, a common generalization of ordinal sum and of direct product. They proved that every BL-algebra embeds into the poset sum of a family of MV-chains and that the embedding is an isomorphism in the finite case. Next, SIU-algebras over arbitrary lattices were classified in [14], see Theorem 4 below. Here the authors assume the existence of a dual-isomorphism between the positive and negative cones of the algebra. For SIU-algebras over weakly real chains, this condition is equivalent to postulating divisibility only for the negative cone of the algebra. In the present paper we classify a class of residuated lattices by assuming only a very weak form of continuity, called absorbent-continuity. It is a much weaker class of residuated lattices by assuming only a very weak form of continuity, called absorbent-continuity. It is a much weaker

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Definition 2: [16] (Twin-rotation construction) Let \( (X_1, \leq) \) be a partially ordered set with top element \( t \) and and \( (X_2, \leq) \) be a partially ordered set with bottom element \( t \) such that the connected ordinal sum \( os_c(X_1, X_2) \) of \( X_1 \) and \( X_2 \) (that is putting \( X_1 \) under \( X_2 \), and identifying the top of \( X_1 \) with the bottom of \( X_2 \)) has an order reversing involution \( i \). Denote the partial order of \( os_c(X_1, X_2) \) also by \( \leq \). Let \( (X_1, \otimes, \oplus) \) and \( (X_2, \otimes, \oplus) \) be commutative semigroups, both with neutral element \( t \). Assume that \( (X_1, \otimes, \oplus) \) is residuated and assume that all residua \( x \to y \) exist if \( x, y \in X_2 \), \( x \leq y \). Assume, in addition, that

1) in case \( t' \in X_1 \), \( x \to t' = x' \) holds for all \( x, x' \in X_1 \), \( x \geq t' \), and

2) in case \( t' \in X_2 \), \( x \to t' = x' \) holds for all \( x, x' \in X_2 \), \( x \leq t' \).

Denote \( U_{\otimes, \oplus} = (os_c(X_1, X_2), \otimes, \leq, t, f) \) where \( f = t' \) and \( \otimes \) is defined as follows:

\[
x \otimes y = \begin{cases} 
  x \otimes y & \text{if } x, y \in X_1 \\
  x \oplus y & \text{if } x, y \in X_2 \\
  (x \to y)' & \text{if } x \in X_2, y \in X_1, \text{ and } x \leq y' \\
  (y \to x)' & \text{if } x \in X_1, y \in X_2, \text{ and } x \leq y' \\
  (x \to (y \land t))' & \text{if } x \in X_2, y \in X_1, \text{ and } x \leq y' \\
  (x \to (y' \land t))' & \text{if } x \in X_1, y \in X_2, \text{ and } x \leq y' 
\end{cases}
\]

Call \( \otimes \) (resp. \( \otimes, \oplus \)) the twin-rotation of \( \otimes \) and \( \oplus \) (resp. of the first and the second partially ordered monoid).

Theorem 1: [16] (Cones representation) Any conic, involutive \( FL_e \)-monoid can be represented as the twin-rotation of its negative and positive cone.

Definition 3: [16] Consider a finite involutive \( FL_e \)-chain \( \mathcal{U} \) and denote the cardinality of its universe by \( n \). Clearly, \( \mathcal{U} \) is isomorphic to a finite involutive \( FL_e \)-chain with the universe \( \{1, 2, \ldots, n\} \subset N \), denote it by \( \mathcal{U}_n = \langle \{1, 2, \ldots, n\}, \cdot, \leq, 1, n, t, f \rangle \). Call \( t - f \) the rank of \( \mathcal{U} \) (or the rank of \( \cdot \)) and denote it by \( \text{rank}(\mathcal{U}) \). It is easy to see that the rank is well-defined. Because of the mentioned order-isomorphism, without loss of generality we will consider finite involutive \( FL_e \)-chains solely on the universe \( \{1, 2, \ldots, n\} \), where \( n \) is any positive integer.

\footnote{This means that for \( x, y \in X_2 \) and \( x \leq y \), the maximal element of the set \( \{z \mid x \oplus z \leq y\} \) exists.}
Let $\mathcal{U} = (X, \ast, \leq, t, f)$ be an involutive FL$_e$-algebra, and $\prime$ be its order-reversing involution. Consider $\mathcal{U}' = (X, \oplus, \leq, f, t)$, where $\oplus$ is the (de Morgan) dual of $\ast$ given by
\[ x \oplus y = (x' \ast y')'. \] (2)

In general, $\mathcal{U}'$ is not an involutive FL$_e$-algebra because $\oplus$ may fail to be residuated (in fact, it will be residuated with respect to the dual ordering relation), and even if it is residuated (for instance, when the underlying universe is a finite chain), it is not involutive.

**Definition 4:** [16] Let $n \geq 1$.
1. Let $\mathcal{U} = \{(1, 2, \ldots, n), \ast, \leq, t, f\}$ be an involutive FL$_e$-algebra with rank $\mathcal{U} > 0$. Define the following algebra: $\mathcal{U}_\lor = \{(1, 2, \ldots, n + 1), \circ, \leq, f + 1, t\}$, where $\circ$ is the dual of $\ast$, and $\circ$ is derived from $\ast$ by adding $n + 1$ as a new annihilator to it. More formally, for $x, y \in \{1, 2, \ldots, n + 1\}$ let
\[ x \circ y = \begin{cases} x \ast y & \text{if } x, y \in \{1, 2, \ldots, n\} \\ n + 1 & \text{if } \max(x, y) = n + 1 \end{cases} \]
and let $x \circ y = (x' \ast y')'$, where $'$ is the order-reversing involution of $\{1, 2, \ldots, n\}$.
2. Let $\mathcal{U} = \{(1, 2, \ldots, n + 1), \ast, \leq, t, f\}$ be an involutive FL$_e$-algebra, and assume $\text{rank}(\mathcal{U}) \leq 0$. Define the following algebra: $\mathcal{U}_\land = \{(1, 2, \ldots, n), \circ, \leq, f, t - 1\}$, where $\circ$ is the restriction of the dual of $\ast$ to $\{1, 2, \ldots, n\}$.

Call $\circ$ the skew dual of $\ast$ both at 1. and at 2. above.

**Theorem 2:** [16] (Finite skew dualization)
1. For any involutive FL$_e$-algebra $\mathcal{U}$ on $\{1, 2, \ldots, n\}$ with rank $k > 0$, $\mathcal{U}_\lor$ is an involutive FL$_e$-algebra on $\{1, 2, \ldots, n + 1\}$ with rank $1 - k$.
2. For any involutive FL$_e$-algebra $\mathcal{U} = \{(1, 2, \ldots, n + 1), \ast, \leq, t, f\}$ with rank $k \leq 0$, $\mathcal{U}_\land$ is an involutive FL$_e$-algebra on $\{1, 2, \ldots, n\}$ with rank $1 - k$.

Moreover,
\[ (\mathcal{U}_\land)_\lor = \mathcal{U} \quad \text{and} \quad (\mathcal{U}_\lor)_\land = \mathcal{U}. \]

**Definition 5:** [16] A finite involutive FL$_e$-algebra $(\{1, 2, \ldots, n\}, \ast, \leq, t, f)$ is $\top$-$\bot$-indecomposable if it has no subalgebra on $\{2, 2, \ldots, n\}$.

**Theorem 3:** [16] (Classification of some classes – finite case) The following statements hold true:
1) (rank 0, rank 1) Let $n \geq 1$. Then, $\ast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank 0 (resp. rank 1) if $n$ is odd (resp. $n$ is even) and
\[ x \ast y = \begin{cases} \min(x, y) & \text{if } x \leq y' \\ \max(x, y) & \text{if } x > y' \end{cases}. \] (3)
2) (rank 2) Let $n \geq 3$ odd. Then, $\ast$ is a $\top$-$\bot$-indecomposable finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank 2 if and only if its underlying t-norm (resp. t-conorm) is $\circ$ on the $\frac{n + 3}{2}$-element chain (resp. an arbitrary t-conorm on the $\frac{n + 2}{2}$-element chain).
3) (rank -1) Let $n$ be an even number such that $n \geq 4$. Then, $\circ$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with the rank $-1$ satisfying $(n - 1) \circ (n - 1) = n$ if and only if its underlying t-norm (resp. t-conorm) is a t-norm $\otimes$ on the $\frac{n}{2}$-element chain satisfying $2 \otimes 2 = 2$ (resp. the dual of $\circ$ on the $\frac{n + 2}{2}$-element chain).
4) (rank $n$) Let $n \geq 2$. Then, $\ast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $3 - n$ if and only if its underlying t-norm (resp. t-conorm) is the (unique) t-norm, namely, the minimum, on the two-element chain (resp. the dual of any Girard monoid on the $n - 1$-element chain).
5) (rank $n$-3) Let $n \geq 3$. Then, $\ast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $n - 3$ if and only if its underlying t-norm satisfies condition 1 in Definition 2 and its underlying t-conorm coincides with the maximum operation on the two-element chain.
6) (rank $5$-n) Let $n \geq 5$. Then, $\ast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $5 - n$ if and only if its underlying t-conorm coincides with the minimum operation on the three-element chain, and its underlying t-norm satisfies condition 2 in Definition 2.

**Theorem 4:** [14] (Classification of SIU-algebras) $\mathcal{U} = (X, \ast, \leq, t, f)$ is a SIU-algebra if and only if its negative cone is a BL-algebra with components which are either cancellative or MV-algebras with two elements, and with no two consecutive cancellative components, $\oplus$ is the dual of $\ast$ with respect to $'$, and $\ast$ is the twin rotation of $\oplus$ and $\ominus$.

**Definition 6:** For an involutive FL$_e$-monoid $\mathcal{U} = (X, \ast, \leq, t, f)$ on a complete poset let
\[ A(x) = \begin{cases} \max\{u \in X^+ \mid u \ast x = x\}, & \text{if } x \in X^+ \\ \inf\{u \in X^- \mid u \ast x = x\}', & \text{if } x \in X^- \end{cases} \]
and call it the absorbent function of $\ast$. The maximum of the set in the first line always exists since $\oplus$ is residuated and the infimum in the second line exists since the poset is complete. Note as well that $A$ is well-defined since $e \in X^- \cap X^+$, $e \ast x = e \ominus x = x$. Call the monoidal operation $\ast$ of a sharp FL$_e$-monoid on a complete poset absorbent-continuous if for $x \in X^-$, $A(x)' \ast x = x$ holds.

**Definition 7:** Call a chain $(X, \leq)$ weakly real if $X$ is order-dense and complete, there exists a dense $Y \subset X$ with $|Y| < |X|$, and for any $x, y \in Y$ there exists $u, v \in Y$ such that $u > x, v > y$, and there exists an order-isomorphism between $[x, u]$ and $[y, v]$.

**Theorem 5:** [15] (Classification of absorbent-continuous, sharp FL$_e$-algebras on weakly real chains) $\mathcal{U} = (X, \ast, \leq, t, f)$ is a absorbent-continuous, sharp FL$_e$-algebra on a weakly real chain if and only if its negative cone is a BL-algebra with components which are either cancellative or MV-algebras with two elements, and with no two consecutive cancellative components, its positive cone is the dual of its negative cone with respect to $'$, and $\ast$ is the twin rotation of $\oplus$ and $\ominus$.

### III. A New Construction
For sharp FL$_e$-chains, the following construction generalizes ordinal sums of Galatos [6, section 3] to the infinite case.

**Definition 8:** Let $(\kappa, \leq)$ be a nonempty totally ordered set. For any $i \in \kappa$ let $\mathcal{U}_i = (X_i, \ast_i, \leq_i, t_i, f_i)$ be a sharp FL$_e$-chain, its involution, its negative and positive cone are being denoted by $'_{i}$, $(X_i^-, \ominus_i)$, and $(X_i^+, \ominus_i)$, respectively. Define
the generalized ordinal sum of the (totally ordered) $X_i$'s as follows:

Let $X = \bigcup_{i \in \kappa} X_i$ be the disjoint union of $X_i$'s equipped by the following total-order:

1) If $x, y \in X_i$ for some $i \in \kappa$ then $x \leq y$ if $x = y$.
2) if $x \in X_i^+$ and $y \in X_j^+$ for some $i, j \in \kappa$ and $i < j$ then $x \leq y$.
3) if $x \in X_i^-$ and $y \in X_j^-$ for some $i, j \in \kappa$ and $i < j$ then $y \leq x$.
4) If $x \in X_i^+$ and $y \in X_j^-$ for some $i, j \in \kappa$ then $x \geq y$.

Identify $U_i$ ($i \in \kappa$) by its embedding into $X$. Notice that 4) implies $t_i = t_j$ for any $i, j \in \kappa$; denote this element by $t$. Then the mapping $(': X \to X, x \mapsto x'$, where $x \in X_i$, is an order-reversing involution of $X$ with $t$ being its fixed point. Define a binary operation $\circ$ on $X$ as follows:

\[
x \circ y = \begin{cases} x \circ_t y & \text{if } x, y \in X_i \\
\min(x, y) & \text{if } x \in X_i, y \in X_j, i \neq j, \text{ and } x \leq y' \\
\max(x, y) & \text{if } x \in X_i, \ \ \ y \in X_j, i \neq j, \ \ \text{and } x > y'
\end{cases}
\]

and call

\[
ios_{i \in \kappa}(U_i) = (X, \circ, \leq, t, t)
\]

the involutive ordinal sum of the family $\mathcal{F} = \{U_i \ | \ i \in \kappa\}$. Further, denote the (max-)ordinal sum of the positive cones of $U_i$'s with respect to $\kappa$ by $(X^+, \oplus)$, that is, let $(X^+, \oplus) = os_{\kappa}(X^+_i, \Theta_i)$. In addition, denote the (min-)ordinal sum of the negative cones of $U_i$'s with respect to the dual of $\kappa$ by $(X^-, \otimes)$, that is, let $(X^-, \otimes) = os_{\kappa}(X^-_i, \Theta_i)$.

**Proposition 6:** The involutive ordinal sum $\circ$ of the family $\mathcal{F} = \{U_i \ | \ i \in \kappa\}$ of sharp FL$_e$-chains $U_i = (X_i, \Theta_i, \leq_t, t_i)$ coincides with the twin-rotation of $\otimes$ and $\oplus$, where $\otimes$ is the (max-)ordinal sum of the positive cones of $U_i$'s with respect to $\kappa$ and $\oplus$ is the (min-)ordinal sum of the negative cones of $U_i$'s with respect to the dual of $\kappa$.

**Theorem 7:** The involutive ordinal sum of an arbitrary family of sharp FL$_e$-chains is a sharp FL$_e$-chain.

**Corollary 8:** The twin-rotation of the Clifford-style ordinal sum of any family of negative cones of sharp FL$_e$-chains is a sharp FL$_e$-chain.

### IV. Conclusion

Classification theorems on residuated monoids have been summarized in this paper along with some new results. The so-called involutive ordinal sum construction have been introduced for constructing sharp FL$_e$-chains. For sharp FL$_e$-chains, the construction generalizes ordinal sums of Galatos [6, section 3] to the infinite case.

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