Uninorms, Nullnorms and Evaluators

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Abstract – At SISY 2009 conference we studied the Sugeno and Shilkret integrals, and some of their level-dependent modifications, as $T$- and/or $S$-evaluators. In this contribution we explore Sugeno and Shilkret integrals, now in connection with uninorms and nullnorms.

Keywords – Sugeno integral, Shilkret integral, uninorm, nullnorm, evaluator.

1 Introduction
1.1 Basic definitions and notation

In [3] we defined an evaluator of elements from a bounded lattice $(L, \wedge, \vee, 1, 0)$ as a monotone mapping $\varphi : L \rightarrow [0, 1]$ satisfying boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 1$. We proposed $T$- and $S$-evaluators for arbitrary t-norm $T$ and t-conorm $S$. We repeat exact definitions later in the text. For details about t-norms and t-conorms we refer to [8].

Clearly, considering $S$-evaluators (for an arbitrary t-conorm $S$) is equivalent to considering a generalized $S$-subadditivity defined by

$$\varphi(a \vee b) \leq S(\varphi(a), \varphi(b))$$

for all $a, b \in L$. For further details we refer to [2,4].

Let $X \neq \emptyset$ be a given (can be uncountable) set. By $\mathcal{F}$ we denote the system of all fuzzy subsets of $X$. Then $(\mathcal{F}, \wedge, \vee, 1, 0)$ is a bounded lattice. We repeat the definition of $T$- and $S$-evaluators from [1], where $T$ and $S$ is a t-norm and a t-conorm, respectively.

Definition 1 A function $\varphi : \mathcal{F} \rightarrow [0, 1]$ is a $T$-evaluator iff

1. $\varphi(1) = 1$, $\varphi(0) = 0$.
2. for all $f, g \in \mathcal{F}$ if $f \leq g$ then $\varphi(f) \leq \varphi(g)$.
3. for all $f, g \in \mathcal{F}$

$$\varphi(f \wedge g) \geq T(\varphi(f), \varphi(g)).$$

A dual evaluator $\varphi^d$ to each evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ is given by

$$\varphi^d(f) = 1 - \varphi(1 - f).$$

Let $\varphi : \mathcal{F} \rightarrow [0, 1]$ be a $T$-evaluator. Then $\varphi^d$ is an $S$-evaluator, i.e., the following holds for each $f, g \in \mathcal{F}$:

$$\varphi^d(f \vee g) \leq S(\varphi^d(f), \varphi^d(g)).$$

In [2] we introduced the notion of a strong $T$-($S$-) evaluator.

Definition 2 Let $T$ and $S$ be a t-norm and a t-conorm, respectively. An evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ is said to be a strong $T$-evaluator if for all $f, g \in \mathcal{F}$

$$\varphi(T(f, g)) \geq T(\varphi(f), \varphi(g)),$$

(1)

and it is said to be a strong $S$-evaluator if

$$\varphi(S(f, g)) \leq S(\varphi(f), \varphi(g)),$$

(2)

where $T(f, g)$ and $S(f, g)$ is meant pointwise.

Obviously, if $\varphi : \mathcal{F} \rightarrow [0, 1]$ is a strong $T$-evaluator and $S$ is the dual t-conorm to $T$ then $\varphi^d$ is a strong $S$-evaluator.

For a finite set $X$, $\varphi(f)$ and $\varphi(g)$ are aggregations of finite inputs. In this case formula (1) is equivalent with the fact that $\varphi$ dominates $T$, and formula (2) is equivalent with the fact that $\varphi$ is dominated by $S$. Domination of aggregation functions was studied in [9,11].

Let $T$ be a t-norm. By $\mu_T$ we denote a fuzzy measure $\mu_T : 2^X \rightarrow [0, 1]$, which is the restriction of a $T$-evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ to crisp sets, only, i.e., to $2^X$. 


Such a measure \( \mu_T \) is called a \( T \)-filter. This means that \( (\mu_T)^d \) is the restriction of the \( S \)-evaluator which is dual to \( \varphi \). Such a measure will be denoted by \( \mu_S \) and called an \( S \)-subadditive measure. \( S \)-subadditive measures were introduced in [2] and further explored in [4,5].

For the purpose of this paper by a measure \( \nu : 2^X \to [0,1] \) we understand a normalized fuzzy measure possessing at least three different values.

Now let us recall the definitions of the Sugeno and Shilkret integrals. More on properties of these integrals can be found in the monographs [7,10].

**Definition 3** Let \( \mu : 2^X \to [0,1] \) be a measure. Mapping \( S : \mathcal{F} \to [0,1] \) defined by:

\[
(S) \int f \, d\mu = \sup_{\alpha \in [0,1]} \min \{ \alpha, \mu(\{ z : f(z) \geq \alpha \}) \} 
\]

for each \( f \in \mathcal{F} \), is called the Sugeno integral with respect to \( \mu \).

**Definition 4** Let \( \mu : 2^X \to [0,1] \) be a measure. Mapping \((Sh) : \mathcal{F} \to [0,1] \) defined by:

\[
(Sh) \int f \, d\mu = \sup_{\alpha \in [0,1]} (\alpha \cdot \mu(\{ z : f(z) \geq \alpha \})) 
\]

for each \( f \in \mathcal{F} \), is called the Shilkret integral with respect to \( \mu \).

### 1.2 Evaluators and fuzzy integrals

Concerning Sugeno integral the following results will be important for our study (see [4]).

**Lemma 1** Let \( \mu \) be an arbitrary measure and \( \nu \) be its dual. Then for all \( f \in \mathcal{F} \) we get

\[
(S) \int f \, d\mu = 1 - (S) \int (1 - f) \, d\nu.
\]

**Theorem 1** Let \( T \) be a t-norm and \( \mu_T : 2^X \to [0,1] \) be a \( T \)-filter. Then \( \varphi(f) = (S) \int f \, d\mu_T \) is a strong \( T \)-evaluator.

**Theorem 2** Let \( S \) be \( t \)-conorm and \( \nu_S : 2^X \to [0,1] \) an \( S \)-subadditive measure. Then \( \psi(f) = (S) \int f \, d\nu_S \) is a strong \( S \)-evaluator.

The Shilkret integral with respect to a measure \( \mu \) is not a \( T_M \)-evaluator when \( \mu \) is a \( T_M \)-filter. However, the following is true (see [2,5]).

**Theorem 3** Let \( T \) be a \( t \)-norm dominated by \( T_P \). Further, let \( \mu_T : 2^X \to [0,1] \) be a \( T \)-filter. Then \( \varphi(f) = (Sh) \int f \, d\mu_T \) is a strong \( T \)-evaluator.

**Definition 5** We say that a \( t \)-conorm \( S \) is subhomogenous if for all \( \alpha, a, b \in [0,1] \)

\[
\alpha S(a, b) \leq S(\alpha a, \alpha b).
\]

An easy computation gives that the four basic \( t \)-conorms, i.e. \( S_M, S_P, S_L, S_D \), are subhomogenous (see [2,5]). However, if we take, e.g.,

\[
S(x, y) = \begin{cases} 1, & \text{if } \max\{x, y\} > 0.3, \\ \max\{x, y\}, & \text{otherwise,} \end{cases}
\]

then, for \( \alpha = x_1 = 0.5 \) we get \( S(\alpha x_1, \alpha x_1) = 0.25 < \alpha S(x_1, x_1) = 0.5 \). Hence \( S \) is not subhomogenous.

In [2,5] conditions were studied under which the Shilkret integral is an \( S \)-evaluator.

**Theorem 4** Let \( S \) be a subhomogenous \( t \)-conorm and \( \nu_S : 2^X \to [0,1] \) be an \( S \)-subadditive measure. Then \( \psi(f) = (Sh) \int f \, d\nu_S \) is an \( S \)-evaluator.

**Lemma 2** Let \( S \) be a \( t \)-conorm that is subhomogenous. Then for an arbitrary \( e \in [0,1] \) the \( t \)-conorm \( S \) which is the ordinal sum \( S = (S, 0, e), (S_M, e, 1) \), is subhomogenous, too.

## 2 \( U \)-Evaluators and \( N \)-Evaluators

### 2.1 Basic definitions

Uninorms and nullnorms (see, e.g. [6]) are associative and commutative operations on \([0,1]\) preserving the natural order. \( U : [0,1]^2 \to [0,1] \) is a uninorm if \( e \in [0,1] \) is its neutral element. If \( U(0,1) = 0 \) we say that \( U \) is conjunctive and if \( U(0,1) = 1 \) we say that \( U \) is disjunctive. \( N : [0,1]^2 \to [0,1] \) is a nullnorm if \( e \in [0,1] \) is its annihilator. If \( U \) is a uninorm then there exists a \( t \)-norm \( T \) and a \( t \)-conorm \( S \) such that

\[
U(x, y) = \begin{cases} e T(\frac{x}{e} \frac{y}{e}), & \text{if } x, y \leq e, \\ (1 - e) S(\frac{x}{1 - e}, \frac{y}{1 - e}) + e, & \text{if } x, y \geq e, \\ A(x, y), & \text{otherwise}, \end{cases}
\]

where \( A : [0,1]^2 \to [0,1] \) is a binary aggregation function satisfying \( \min(x, y) \leq A(x, y) \leq \max(x, y) \). Uninorm \( U \) defined by (5), will be denoted as \( U = \langle T, S, e, \min \rangle \) or \( U = \langle T, S, e, \max \rangle \), if \( A = \min \) or \( A = \max \), respectively.
If $N$ is a nullnorm then there exist a t-norm and a t-conorm $S$ such that
\[
N(x, y) = \begin{cases} 
  eS(\frac{x}{e}, \frac{y}{e}), & \text{if } x, y \leq e, \\
  (1 - e)T(\frac{x}{1-e}, \frac{y}{1-e}) + e, & \text{if } x, y \geq e, \\
  e, & \text{otherwise.} 
\end{cases} 
\]
(6)
A nullnorm $N$ defined by (6) will be denoted as $N = \langle S, T, e \rangle$. For properties of uninorms and nullnorms we refer to [6].

$U$-evaluators were introduced in [3] and their construction via level-dependent Sugeno integrals was studied. Now a slightly modified definition of $U$-evaluators will be presented, and strong $U$-evaluators and strong $N$-evaluators will be introduced. Also possibilities of construction of these evaluators via Sugeno and Shilkret integrals will be investigated.

**Definition 6** Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm and $e \in [0, 1]$ be its neutral element. We say that an evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ is a $U$-evaluator if
\[
\varphi(f \wedge g) \begin{cases} 
  \geq U(\varphi(f), \varphi(g)), & \text{if } \max\{\varphi(f), \varphi(g)\} \leq e, \\
  \leq U(\varphi(f), \varphi(g)), & \text{otherwise}, \\
\end{cases}
\]
\[
\varphi(f \vee g) \begin{cases} 
  \leq U(\varphi(f), \varphi(g)), & \text{if } \min\{\varphi(f), \varphi(g)\} \geq e, \\
  \geq U(\varphi(f), \varphi(g)), & \text{otherwise}. 
\end{cases}
\]

**Definition 7** Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm and $e \in [0, 1]$ be its neutral element. An evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ is said to be a strong $U$-evaluator if
\[
\varphi(U(f, g)) \begin{cases} 
  \geq U(\varphi(f), \varphi(g)), & \text{if } \max\{\varphi(f), \varphi(g)\} \leq e, \\
  \leq U(\varphi(f), \varphi(g)), & \text{if } \min\{\varphi(f), \varphi(g)\} \geq e, \\
  = U(\varphi(f), \varphi(g)), & \text{otherwise}. 
\end{cases}
\]

**Definition 8** Let $N : [0, 1]^2 \rightarrow [0, 1]$ be a nullnorm and $e \in [0, 1]$ be its annihilator. An evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ is said to be an $N$-evaluator if
\[
\varphi(f \wedge g) \begin{cases} 
  \geq N(\varphi(f), \varphi(g)), & \text{if } \min\{\varphi(f), \varphi(g)\} \geq e, \\
  \leq N(\varphi(f), \varphi(g)), & \text{otherwise}, \\
\end{cases}
\]
\[
\varphi(f \vee g) \begin{cases} 
  \leq N(\varphi(f), \varphi(g)), & \text{if } \max\{\varphi(f), \varphi(g)\} \leq e, \\
  \geq N(\varphi(f), \varphi(g)), & \text{otherwise}. 
\end{cases}
\]

**Definition 9** Let $N : [0, 1]^2 \rightarrow [0, 1]$ be a nullnorm and $e \in [0, 1]$ be its annihilator. An evaluator $\varphi : \mathcal{F} \rightarrow [0, 1]$ is said to be a strong $N$-evaluator if
\[
\varphi(N(f, g)) \begin{cases} 
  \leq N(\varphi(f), \varphi(g)), & \text{if } \max\{\varphi(f), \varphi(g)\} \leq e, \\
  \geq N(\varphi(f), \varphi(g)), & \text{if } \min\{\varphi(f), \varphi(g)\} \geq e, \\
  = e, & \text{otherwise.} 
\end{cases}
\]

2.2 Construction of $U$-evaluators and $N$-evaluators

An obvious possibility how strong $U$- and $N$-evaluators can be constructed is presented in the following example.

**Example 1** Let $X$ be an $n$-element set, $U$ be a uninorm and $N$ a nullnorm. Define $\eta : \mathcal{F} \rightarrow [0, 1]$ and $E : \mathcal{F} \rightarrow [0, 1]$ by
\[
\eta(f) = U(f(x_1), f(x_2), \ldots, f(x_n)), \\
E(f) = N(f(x_1), f(x_2), \ldots, f(x_n)),
\]
for an arbitrary $f \in \mathcal{F}$. Then $\eta$ is a strong $U$-evaluator and $E$ is a strong $N$-evaluator.

Next we show some constructions of (strong) $U$-evaluators and (strong) $N$-evaluators using Sugeno and Shilkret integrals.

**Notation** Let $T$ be an arbitrary t-norm, $S$ an arbitrary t-conorm and $e \in [0, 1]$. Further, let $f \in \mathcal{F}$. Then
\[
\tilde{T} = (\langle T, 0, e \rangle), \\
\tilde{S} = (\langle S, 0, e \rangle), \\
\mu_{\tilde{T}} : 2^X \rightarrow [0, 1] \text{ and } \mu_{\tilde{S}} : 2^X \rightarrow [0, 1] \text{ be arbitrarily chosen } \tilde{T} \text{- and } \tilde{S} \text{-filters, respectively.}
\]
\[
\nu_{\tilde{S}} : 2^X \rightarrow [0, 1] \text{ and } \nu_{\tilde{S}} : 2^X \rightarrow [0, 1] \text{ be arbitrarily chosen } \tilde{S} \text{- and } \tilde{S} \text{-subadditive measures.}
\]
\[
\sigma_{\mu_{\tilde{T}}}(f) = (S) \int f d\mu_{\tilde{T}}, \\
\nu_{\tilde{S}}(f) = (S) \int f d\nu_{\tilde{S}}, \\
\sigma_{\nu_{\tilde{S}}}(f) = (S) \int f d\nu_{\tilde{S}}, \\
\theta_{\nu_{\tilde{S}}}(f) = (S) \int f d\nu_{\tilde{S}}.
\]
• $\Xi_{\psi_{SM}}(f) = (Sh) \int f \, dv_{\psi_{SM}}$
• $\xi_{\mu_{TM}}(f) = \left( \Xi_{\psi_{SM}}(f) \right)^d$
where $\psi_{SM}$ and $\mu_{TM}$ are dual maps for $(max, \psi)$ and $(min, \nu)$, respectively.

• $\Omega_{\psi_{S}}(f) = (Sh) \int f \, dv_{\psi_{S}}$
• $\omega_{\mu_{T}}(f) = \left( \Omega_{\psi_{S}}(f) \right)^d$
where $\psi_{S}$ and $\mu_{T}$ are dual maps for the $t$-norm and $t$-conorm, respectively.

**Theorem 5** Let $e \in ]0, 1[\rightarrow [0, 1]$ be an arbitrary uninorm such that
$$U(x,y) = \begin{cases} T(x,y), & \text{if } \max\{x,y\} \leq e, \\ S(x,y), & \text{if } \min\{x,y\} \geq e. \end{cases}$$
Then $\phi : \mathcal{F} \rightarrow [0, 1]$ and $\psi : \mathcal{F} \rightarrow [0, 1]$, defined as
$$\phi(f) = \begin{cases} \sigma_{\mu_{T}}(f), & \text{if } \sigma_{\mu_{T}}(f) \leq e, \\ \max\{e, \sigma_{\psi_{S}}(f)\}, & \text{otherwise,} \end{cases}$$
$$\psi(f) = \begin{cases} \sigma_{\psi_{S}}(f), & \text{if } \sigma_{\psi_{S}}(f) \geq e, \\ \min\{e, \sigma_{\mu_{T}}(f)\}, & \text{otherwise,} \end{cases}$$
are $U$-evaluators.

**Theorem 6** Let $T$ and $S$ be arbitrary $t$-norm and $t$-conorm, respectively and $e \in [0, 1]$. Then for the uninorm $U = (T,S,e,\min)$, the $U$-evaluator $\phi$, defined by (7), is a strong $U$-evaluator.

For the uninorm $U = (T,S,e,\max)$, the $U$-evaluator $\psi$, defined by (8), is a strong $U$-evaluator.

**Example 2** Let us consider a set $X = \{x_1, \ldots, x_6\}$, a $t$-norm and a $t$-conorm respectively $T = (\langle T_p, 0, 1/2, \rangle, \langle T_m, 1/2, 1 \rangle)$, $S = (\langle S_m, 0, 1/2, \rangle, \langle S_l, 1/2, 1 \rangle)$, measure $\mu_T$
$$\mu_T(A) = \begin{cases} 1/3, & \text{if } A = \{x_1\}, \\ 1/3, & \text{if } A = \{x_1, x_2\} \text{ or } A = \{x_1, x_3\}, \\ 1/2, & \text{if } A = \{x_1, x_2, x_3\}, \\ 3/4, & \text{if } \{x_1, x_2, x_3, x_4\} \subset A \neq X, \\ 1, & \text{if } A = X, \\ 0, & \text{otherwise,} \end{cases}$$
and a $\tilde{S}$-subadditive measure $v_{\tilde{S}}$ given by the following system of weights
$$w_i = \begin{cases} 1/2, & \text{for } i = 1, 3, \\ 3/4, & \text{for } i = 4, \\ 5/8, & \text{for } i = 5, 6, \\ 1/6, & \text{for } i = 1, 3. \end{cases}$$

We take evaluators $\phi$ and $\psi$, given by (7) and (8), respectively. These are $U$-evaluators for both uninorms, $U_1 = (T_p, S_l, 1/2, \min)$ and $U_2 = (T_p, S_l, 1/2, \max)$. Let $f, g \in \mathcal{F}$ be functions
$$f(x) = \begin{cases} 2/3, & \text{for } x \in \{x_1, x_2\}, \\ 0, & \text{otherwise,} \end{cases}$$
$$g(x) = \begin{cases} 1/3, & \text{for } x \in \{x_2, x_3, x_4\}, \\ 0, & \text{otherwise.} \end{cases}$$
Then we have
$$\langle f \wedge g \rangle(x) = \begin{cases} 1/3, & \text{for } x = x_2, \\ 0, & \text{otherwise,} \end{cases}$$
$$\langle f \vee g \rangle(x) = \begin{cases} 3/4, & \text{for } x \in \{x_2, x_3, x_4\}, \\ 1/4, & \text{for } x = x_1, \\ 0, & \text{otherwise.} \end{cases}$$
From this we get $\phi(f) = \psi(f) = 1/2$, $\phi(g) = 0$, $\psi(g) = 3/4$, $\psi(U_1(f,g)) = \phi(\langle f \wedge g \rangle) = 0$, $\phi(\langle f \vee g \rangle) = \phi(U_2(f,g)) = 3/4$, $\psi(\langle f \vee g \rangle) = \psi(U_2(f,g)) = 3/4$. This means that $\phi$ is a strong $U_1$-evaluator but it is not a strong $U_2$-evaluator, and $\psi$ is a strong $U_2$-evaluator but it is not a strong $U_1$-evaluator.

**Theorem 7** Let $\mu_{TM} : 2^X \rightarrow [0, 1]$ and $\nu_{SM} : 2^X \rightarrow [0, 1]$ be respectively minitive and maxitive measures, and let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm such that for an $e \in [0, 1]$ we have
$$U(x,y) = \begin{cases} T_M(x,y), & \text{if } \max\{x,y\} \leq e, \\ S_M(x,y), & \text{if } \min\{x,y\} \geq e. \end{cases}$$
Then $\Phi : \mathcal{F} \rightarrow [0, 1]$ and $\Psi : \mathcal{F} \rightarrow [0, 1]$, defined as
$$\Phi(f) = \begin{cases} \xi_{\mu_{TM}}(f), & \text{if } \xi_{\mu_{TM}}(f) \leq e, \\ \max\{e, \Xi_{\psi_{SM}}(f)\}, & \text{otherwise,} \end{cases}$$
$$\Psi(f) = \begin{cases} \Xi_{\psi_{SM}}(f), & \text{if } \Xi_{\psi_{SM}}(f) \geq e, \\ \min\{e, \xi_{\mu_{TM}}(f)\}, & \text{otherwise.} \end{cases}$$
are $U$-evaluators. Moreover, if $U = (T_M, S_M, e, \min)$ then $\Phi$ is a strong $U$-evaluator, and if $U = (T_M, S_M, e, \max)$ then $\Psi$ is a strong $U$-evaluator.

**Theorem 8** Let $T$ and $S$ be arbitrary $t$-norm and $t$-conorm, respectively and $e \in [0, 1]$. Further, let $N$ be the nullnorm $N = (S, T, e)$. Then $\gamma : \mathcal{F} \rightarrow [0, 1]$ and
\( \delta : \mathcal{F} \rightarrow [0, 1] \) defined respectively by

\[
\gamma(f) = \begin{cases} \theta_{\nu_2}(f), & \text{if } \theta_{\nu_2}(f) \leq e, \\ \max\{e, \theta_{\mu_2}(f)\}, & \text{otherwise,} \end{cases}
\]

(9)

\[
\delta(f) = \begin{cases} \theta_{\mu_2}(f), & \text{if } \theta_{\mu_2}(f) \geq e, \\ \min\{e, \theta_{\nu_2}(f)\}, & \text{otherwise,} \end{cases}
\]

(10)

are \( N \)-evaluators. Moreover, both \( \delta \) and \( \gamma \) are strong \( N \)-evaluators.

**Example 3** Let us consider the nullnorm \( N = \langle S_L, T_P, \frac{1}{2} \rangle \), the t-norm \( T = (T_M, 0, \frac{1}{2}, \frac{1}{2}, 1) \) and the t-conorm \( S = (S_L, 0, \frac{1}{2}, (S_M, \frac{1}{2}, 1)) \). Let us use the set \( X \) from Example 2. Take \( \nu_2 \) an \( S \)-subadditive measure given by the following weights

\[
w_i = \begin{cases} \frac{1}{3}, & \text{for } i = 1, 2, 3, 4, \\ \frac{3}{4}, & \text{for } i = 6, \\ 1, & \text{for } i = 5, \end{cases}
\]

and \( \mu_{\bar{P}} \) the following \( \bar{P} \)-filter

\[
\mu_{\bar{P}}(A) = \begin{cases} \frac{1}{3}, & \text{if } x_6 \in A \subseteq \{x_4, x_5, x_6\}, |A| \leq 2, \\ \frac{1}{4}, & \text{if } A = \{x_4, x_5, x_6\}, \\ \frac{3}{4}, & \text{if } x_1 \in A, x_2, x_3 \notin A, \\ \frac{2}{7}, & \text{if } \{x_1, x_2\} \subseteq A, x_3 \notin A, \\ 1, & \text{if } \{x_1, x_2, x_3\} \subseteq A, \\ 0, & \text{otherwise.} \end{cases}
\]

Let \( f, g \in \mathcal{F} \) be functions

\[
f(x) = \begin{cases} \frac{5}{8}, & \text{for } x \in \{x_1, x_2, x_3\}, \\ 0, & \text{otherwise,} \end{cases}
\]

\[
g(x) = \begin{cases} \frac{3}{4}, & \text{for } x \in \{x_1, x_4, x_6\}, \\ 0, & \text{otherwise.} \end{cases}
\]

Then, for \( \gamma \) and \( \delta \) given by (9) and (10), respectively, we have

\[
\gamma(f) = \frac{3}{8}, \quad \delta(f) = \frac{5}{8}, \quad \gamma(g) = \delta(g) = \frac{3}{4}, \quad \gamma(N(f, g)) = \frac{1}{2} = \gamma(\gamma(f), \gamma(g)), \quad \delta(N(f, g)) = \frac{3}{16} = \delta(\delta(f), \delta(g)).
\]

Both, \( \gamma \) and \( \delta \) are strong \( N \)-evaluators.

**Theorem 9** Let \( S \) be a given subhomogenous t-conorm and \( T \) be an arbitrary t-norm that is dual to a subhomogenous t-conorm, and \( e \in [0, 1] \). For the nullnorm \( N = \langle S, T, e \rangle \) we obtain that \( \Gamma : \mathcal{F} \rightarrow [0, 1] \) and \( \Delta : \mathcal{F} \rightarrow [0, 1] \) defined respectively by

\[
\Gamma(f) = \begin{cases} \Omega_{\nu_2}(f), & \text{if } \Omega_{\nu_2}(f) \leq e, \\ \max\{e, \omega_{\mu_2}(f)\}, & \text{otherwise,} \end{cases}
\]

(11)

\[
\Delta(f) = \begin{cases} \omega_{\mu_2}(f), & \text{if } \omega_{\mu_2}(f) \geq e, \\ \min\{e, \Omega_{\nu_2}(f)\}, & \text{otherwise,} \end{cases}
\]

(12)

are \( N \)-evaluators.

**Theorem 10** Let \( N = \langle S_M, T_M, e \rangle \) for \( e \in [0, 1] \). Further, let \( \nu_{S_M} \) and \( \mu_{T_M} \) be maxitive and minitive measures respectively. Then

\[
\Gamma(f) = \begin{cases} \Omega_{\nu_{S_M}}(f), & \text{if } \Omega_{\nu_{S_M}}(f) \leq e, \\ \max\{e, \omega_{\mu_{T_M}}(f)\}, & \text{otherwise,} \end{cases}
\]

\[
\Delta(f) = \begin{cases} \omega_{\mu_{T_M}}(f), & \text{if } \omega_{\mu_{T_M}}(f) \geq e, \\ \min\{e, \Omega_{\nu_{S_M}}(f)\}, & \text{otherwise,} \end{cases}
\]

are strong \( N \)-evaluators.

**3 Conclusions**

In this paper we discussed conditions under which the Sugeno and Shilkret integrals are (strong) \( U \)- or (strong) \( N \)-evaluators. These evaluators consist always of two parts. Below and above the neutral element if considering (strong) \( U \)-evaluators, and below and above the annihilator if considering (strong) \( N \)-evaluators. In one part they behave like (strong) \( T \) and in the other part like (strong) \( S \)-evaluators.

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**References**


