Jensen type inequality for the pseudo-integral based on the smallest universal integral

Endre Pap *,†, Mirjana Štroba ‡,
* Singidunum University, Trg D. Obradovica 4, Novi Sad, Serbia
† Óbuda University, Becsi út 96/B, H-1034 Budapest, Hungary
‡ Faculty of Sciences, University of Novi Sad, Trg D. Obradovica 4, Novi Sad, Serbia
E-mails: pape@eunet.rs; epap@singidunum.ac.rs; mirjana.strboja@dmi.uns.ac.rs

Abstract—Choquet and Sugeno integrals have wide applications in several practical areas, especially as aggregation functions in decision theory. Universal integral is a generalization of Choquet and Sugeno integrals. The Jensen type inequality related to the smallest universal integral which special cases are Sugeno, Shilkret, seminormed fuzzy integrals and pseudo-integral has been recently proposed. Based on this result, we have obtained the Jensen’s inequality for the pseudo-integral and sufficient conditions under which that inequality holds.

Keywords. Pseudo-operation, monotone measure, universal integral, pseudo-integral, Jensen’s inequality.

I. INTRODUCTION

Non-additive measure, called also fuzzy measure or capacity, and the corresponding integrals, e.g., Choquet, Sugeno, idempotent integrals, see [11], [16], [18], [27], [28], [37], [40], have the advantage with respect to additive measure and corresponding integrals based on them, deal with modeling of certain phenomena involving interaction between criteria. Choquet and Sugeno integrals have wide applications as aggregation functions, see [14], [35], [38], [40]. Recently, the universal integral, whose special cases are all the mentioned integrals, has been proposed in [17].

Inequalities for Choquet, Sugeno, seminormed fuzzy integrals and pseudo-integral have been recently proposed, see [1], [2], [3], [4], [5], [6], [12], [13], [19], [20], [23], [24], [29], [31]. Román-Flores et al. considered the Jensen type inequality for Sugeno integral in [31]. A fuzzy Chebyshev type inequality has been studied in [2], [12], [20], [23], [25]. The properties of Choquet integral and related inequalities are observed by Wang [41] and Mesiar, Li, and Pap [19]. Hölder and Minkowski type inequalities for universal integral is given in [6].

In the classical measure theory the Jensen inequality for Lebesgue integral is concerned with convex functions, see [33]. A Jensen type inequality for the special cases of the universal integral is valid for a wider class of functions, i.e., subhomogeneous functions, see [30].

Our paper is organized as follows. In Section II are given definitions of the classes of functions used in this paper and the relationship among these classes of functions. There are given also some preliminaries on the monotone measure and universal integral. As a special cases of the smallest universal integral we recall definitions of Sugeno, Shilkret integrals and pseudo-integral in Section II-B. We give an overview of generalizations of the Jensen’s inequality for the smallest universal integral and its corollaries related to Sugeno and Shilkret integrals in Section III. In Section IV we obtain a sufficient condition for the Jensen’s inequality related to the pseudo-integral.

II. PRELIMINARY NOTIONS

A. Some classes of functions and the universal integral

Throughout this paper we are considering the convex, subhomogeneous and superadditive functions ([8]).

Definition 1: If a function $f : [0, \infty[ \to [0, \infty[\text{ satisfies inequality}$

i) $f (\lambda x + (1 - \lambda) y) \leq \lambda f (x) + (1 - \lambda) f (y)$

for all $\lambda \in [0, 1]$ and $x \geq 0$, $y \geq 0$, then it is convex,

ii) $f (\lambda x) \leq \lambda f (x)$

for all $\lambda \in [0, 1]$ and $x \geq 0$, then it is subhomogeneous,

iii) $f (x + y) \geq f (x) + f (y)$

for all $x, y \in [0, \infty[$, then it is superadditive.

Let $C, \mathcal{H}$ and $\mathcal{D}$ be the classes of convex, subhomogeneous and superadditive functions, respectively, which are continuous, nonnegative and for which $f (0) = 0$. The relationship between these three classes of functions has been studied in [8]. Namely, the following inclusions are valid but the reverse inclusions does not hold:

$$\mathcal{C} \subset \mathcal{H} \subset \mathcal{D}. \tag{1}$$

The following result is very useful ([8]).

Lemma 2: A function $f : [0, \infty[ \to [0, \infty[\text{ is subhomoge-

neous iff } f(x) x}$ is an increasing function.

Let $(X, \mathcal{A})$ be a measurable space.

Definition 3: ([17], [27], [37]) If a function $m : \mathcal{A} \to [0, \infty]$ satisfies:

i) $m (\emptyset) = 0$,

ii) $m (X) > 0$,

iii) for all $A, B \in \mathcal{A}$ if $A \subseteq B$, then $m (A) \leq m (B)$,

then $m$ is a monotone measure on a measurable space $(X, \mathcal{A})$. 
Recall that a function \( m : \mathcal{A} \rightarrow [a, b] \) is a sup-measure if for any system \((A_i)_{i \in I}\) of measurable sets,
\[
m \left( \bigcup_{i \in I} A_i \right) = \sup_{i \in I} m(A_i).
\]
We shall consider complete sup-measure \( m \) of the form
\[
m(A) = \sup_{x \in A} \psi(x),
\]
where \( \psi : X \rightarrow [a, b] \) is a density function and \([a, b]\) is a closed subinterval of \([-\infty, +\infty]\). If \( \mathcal{A} = 2^X \) then any complete sup-measure is of the form \( m(A) = \sup_{x \in A} \psi(x) \), where \( \psi : X \rightarrow [a, b] \) is a density function (profile) given by \( \psi(x) = m(\{x\}) \). Note that for \([a, b] = [0, \infty]\) complete sup-measure is monotone measure in sense of Definition 3.

A function \( f : X \rightarrow [0, \infty] \) is \( \mathcal{A} \)-measurable if for each \( B \in \mathcal{B}([0, \infty]) \) holds \( f^{-1}(B) \in \mathcal{A} \), where \( \mathcal{B}([0, \infty]) \) is the \( \sigma \)-algebra of Borel subsets of \([0, \infty]\). The following notations are used:
- \( \mathcal{F}(X, \mathcal{A}) \) denote the set of all \( \mathcal{A} \)-measurable functions \( f : X \rightarrow [0, \infty] \);
- For each number \( a \in [0, \infty] \), \( \mathcal{M}_a(X, \mathcal{A}) \) is the set of all monotone measures satisfying \( m(X) = a \), and denote by \( \mathcal{M}(X, \mathcal{A}) = \bigcup_{a \in [0, \infty]} \mathcal{M}_a(X, \mathcal{A}) \);
- \( S \) is the class of all measurable spaces, and \( D_{[0, \infty]} = \bigcup_{(X, \mathcal{A}) \in S} \mathcal{F}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A}) \).

An equivalence relation between pairs \((m_1, f_1), (m_2, f_2) \in D_{[0, \infty]} \) was introduced in [17].

**Definition 4:** Two pairs \((m_1, f_1) \in \mathcal{M}(X_1, \mathcal{A}_1) \times \mathcal{F}(X_1, \mathcal{A}_1) \) and \((m_2, f_2) \in \mathcal{M}(X_2, \mathcal{A}_2) \times \mathcal{F}(X_2, \mathcal{A}_2) \) satisfying
\[
m_1(\{x \in X_1 \mid f_1(x) \geq t\}) = m_2(\{x \in X_2 \mid f_2(x) \geq t\})
\]
for all \( t \in [0, \infty] \), are integral equivalent.

The notion of operation is important for introducing the universal integral.

**Definition 5:** ([17], [37]) A pseudo-multiplication is a function \( \otimes : [0, \infty]^2 \rightarrow [0, \infty] \) with the following properties:

i) it is non-decreasing in each component, i.e., for all \( a_1, a_2, b_1, b_2 \in [0, \infty] \) with \( a_1 \leq a_2 \) and \( b_1 \leq b_2 \) we have \( a_1 \otimes b_1 \leq a_2 \otimes b_2 \),

ii) 0 is a neutral element of \( \otimes \), i.e., for all \( a \in [0, \infty] \) we have \( a \otimes 0 = 0 \otimes a = 0 \),

iii) it has a neutral element different from 0, i.e., there exists an \( e \in [0, \infty] \) such that, for all \( a \in [0, \infty] \) we have \( a \otimes e = e \otimes a = a \).

In general, a pseudo-multiplication is neither a commutative nor associative operation.

The universal integral is introduced by axioms ([17]).

**Definition 6:** A function \( I : D_{[0, \infty]} \rightarrow [0, \infty] \) is a universal integral if the following axioms hold:

i) the restriction of the function \( I \) to \( \mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A}) \) is non-decreasing in each coordinate for any measurable space \((X, \mathcal{A})\);

ii) there is a pseudo-multiplication \( \otimes : [0, \infty]^2 \rightarrow [0, \infty] \) such that for all pairs \((m, c \cdot 1_A) \in D_{[0, \infty]} \) (where \( 1_A \) is the characteristic function of the set \( A \))
\[
I(m, c \cdot 1_A) = c \otimes m(A);
\]

iii) it holds
\[
I(m_1, f_1) = I(m_2, f_2),
\]
for all integral equivalent pairs \((m_1, f_1), (m_2, f_2) \in D_{[0, \infty]} \).

In the following proposition the smallest and the greatest universal integral are given based on \( \otimes \) ([17]).

**Proposition 7:** The smallest universal integral \( I_s \) and the greatest universal integral \( I^\oplus \) based on a pseudo-multiplication \( \otimes \) on \([0, \infty]\) are given by
\[
I_s(m, f) = \sup_{t \in [0, \infty]} \left( t \otimes m(\{x \in X \mid f(x) \geq t\}) \right), \quad (3)
\]
\[
I^\oplus(m, f) = \operatorname{esssup}_m f \otimes \sup_{t \in [0, \infty]} m(\{x \in X \mid f(x) \geq t\}),
\]
where
\[
\operatorname{esssup}_m f = \sup\{t \in [0, \infty] \mid m(\{x \in X \mid f(x) \geq t\}) > 0\}.
\]

**B. Special cases of the smallest universal integral**

The Sugeno [37] and Shilkret [34] integral, respectively, are given for any measurable space \((X, \mathcal{A})\) for any monotone measure \( m \in \mathcal{M}(X, \mathcal{A}) \), by
\[
\operatorname{Su}(m, f) = \sup_{t \in [0, \infty]} \left( \min(t, m(\{x \in X \mid f(x) \geq t\})) \right),
\]
\[
\operatorname{Sh}(m, f) = \sup_{t \in [0, \infty]} \left( t \cdot m(\{x \in X \mid f(x) \geq t\}) \right),
\]
respectively, where the convention \( 0 \cdot \infty = 0 \) is used.

For the pseudo-multiplication \( \otimes = \min \) and \( \otimes = \cdot \), the smallest universal integral given by (3) reduces to the Sugeno and Shilkret integral, i.e., \( I_s = I_{\min} \) and \( I^\oplus = I \), respectively.

Let \([a, b]\) be a closed subinterval of \([-\infty, +\infty]\). If the operation \( \otimes : [a, b]^2 \rightarrow [a, b] \) is defined by an increasing and continuous function \( g : [a, b] \rightarrow [0, \infty] \), such that \( x \otimes y = g^{-1}(g(x)g(y)) \), \( x, y \in [a, b] \), then the pseudo-integral Related to the semiring \([a, b], \sup, \otimes \) (see [27]), for a measurable function \( f : X \rightarrow [a, b] \) is given by
\[
\operatorname{Ps}(m, f) = \sup_{x \in X} \left( f(x) \otimes \psi(x) \right),
\]
where function \( \psi \) defines sup-measure \( m \) by (2), for more details see [18], [26], [27], [28]. The pseudo-integral Related to the semiring \([0, \infty], \sup, \otimes \) is the smallest universal integral based on the pseudo-multiplication \( \otimes \) on \([0, \infty] \).
As a corollary of the results from [29] a Jensen type inequality for the pseudo-integral holds.

**Theorem 8:** Let \( \odot : [0, \infty]^2 \to [0, \infty] \) be a pseudo-multiplication on \([0, \infty]\) given by \( x \odot y = g^{-1}(g(x)g(y)) \), where \( g \) be a strictly increasing, convex bijection. Let \( \lambda \) be the usual Lebesgue measure on \( \mathbb{R} \) and \( m \) be a sub-measure on \( ([c, d], \mathcal{B}([c, d])) \) such that
\[
m(A) = \text{esssup}_y (\psi(x) \mid x \in A),
\]
where \( \psi : [c, d] \to [0, \infty] \) is a continuous density and \( m([c, d]) = 1 \) and \( e \) is the neutral element of \( \odot \). If \( f : [c, d] \to [0, \infty] \) is continuous function such that pseudo-integral \( P_s(f, m, \phi) \) finite and \( \phi \) is a convex and non-increasing function on the range of \( f \), then, it holds:
\[
\phi(P_s(m, f)) \leq P_s(m, \phi(f)).
\]

The smallest universal integral on the interval \([0, 1]\) related to the restriction of pseudo-multiplication to \([0, 1]\) is a type of integral which is called seminormed integral in [36].

### III. JENSEN’S INEQUALITY FOR THE SMALLEST UNIVERSAL INTEGRAL

A Jensen type inequality related to the smallest universal integral is given in [30]. In a special case the following theorem reduces to the corresponding inequality for Sugeno integral obtained in [31].

**Theorem 9:** Let \( \odot : [0, \infty]^2 \to [0, \infty] \) be a pseudo-multiplication on \([0, \infty]\). If \( \phi : [0, \infty] \to [0, \infty] \) is continuous, strict increasing function and
\[
\phi(x \odot y) \leq \phi(x) \odot y
\]
for all \( y \in [0, m(X)] \) and \( x \in [0, \infty] \), then
\[
\phi(I_0 (m, f)) \leq I_0 (m, \phi(f)).
\]
for all \( f \in \mathcal{F}(X,A) \) and \( m \in \mathcal{M}(X,A) \).

If function \( \phi \) satisfies \( \phi(x) \leq x \) for every \( x \in [0, m(X)] \) and \( \odot = \min \) in Theorem 9, the condition (4) holds. Also, if \( \odot = \min \) and the condition (4) holds then \( \phi(x) \leq x \) for every \( x \in [0, m(X)] \). Therefore, the inequality (5) reduces to the Jensen’s inequality for Sugeno integral which was obtained in [31].

**Corollary 10:** If \( \phi : [0, \infty] \to [0, \infty] \) is continuous and strict increasing function such that \( \phi(x) \leq x \) for every \( x \in [0, m(X)] \), then the following holds
\[
\phi(Su(m, f)) \leq Su(m, \phi(f))
\]
for all \( f \in \mathcal{F}(X,A) \).

If the pseudo-multiplication is given by \( \odot = \cdot \), then the inequality (5) is Jensen’s inequality for Shilkret integral ([30]).

**Corollary 11:** If \( \phi : [0, \infty] \to [0, \infty] \) is continuous, strict increasing and subhomogeneous function, then it holds
\[
\phi(Sh(m, f)) \leq Sh(m, \phi(f))
\]
for all \( f \in \mathcal{F}(X,A) \) and \( m \in \mathcal{M}_1(X,A) \).

### IV. JENSEN’S INEQUALITY FOR PSEUDO-INTEGRAL

We shall consider pseudo-multiplication \( \odot \) generated by an increasing bijection \( g : [0, \infty] \to [0, \infty] \), i.e. \( x \odot y = g^{-1}(g(x)g(y)) \), \( x, y \in [0, \infty] \). Observe that \( g(0) = 0 \) and \( e = g^{-1}(1) \). Due to Theorem 9 we have the Jensen’s inequality for pseudo-integral based on a complete sub-measure, i.e., the following theorem holds.

**Theorem 12:** Let \( \odot : [0, \infty]^2 \to [0, \infty] \) be a pseudo-multiplication represented by an increasing generator \( g : [0, \infty] \to [0, \infty] \), \( m \) be a complete sub-measure and \( m(X) = e \), where \( e \) is the neutral element of \( \odot \). Let \( \phi \) be continuous, strictly increasing function such that \( g \circ \phi \circ g^{-1} \) is a subhomogeneous function. Then for any \( f \in \mathcal{F}(X,A) \) we have
\[
\phi(P_s(m, f)) \leq P_s(m, \phi(f)).
\]

**Remark 13:** The condition of subhomogeneity of \( g \circ \phi \circ g^{-1} \) in Theorem 12 is satisfied if either \( \frac{\psi(x)}{g(x)} \) is an increasing function or \( g \circ \phi \circ g^{-1} \) is a convex function and \( \phi(0) = 0 \).

**Example 14:** Let \( \phi(x) = e^{2x} - 1 \), \( g(x) = e^x - 1 \), \( x \in [0, \infty] \) then we have
\[
g(\phi^{-1}(x)) = e^{2x} + x^2 - 1.
\]
The composition \( g \circ \phi \circ g^{-1} \) is a convex function. Therefore, we have
\[
e^{2 \left( \sup_{x \in X} f(x) \circ \phi(x) \right)} - 1 \leq \sup_{x \in X} \left( e^{2f(x) - 1} \circ \phi(x) \right),
\]
where function \( \psi \) defines sub-measure \( m \), \( m(X) = e = \ln 2 \) and \( x \odot y = \ln (e^x e^y - e^y - e^x + 2) \).

Now let us consider the sufficient condition for \( g \circ \phi \circ g^{-1} \) to be subhomogeneous function. A generalization of the classical convexity of function has been introduced by many authors ([10], [21], [22], [39]). In this paper we shall use the following type of convexity ([9]).

**Definition 15:** Let \( I \subseteq \mathbb{R} \) be an interval. If \( f, g : I \to \mathbb{R} \) are continuous functions and \( g \) is strictly monotonic, then \( f \) is convex with respect to \( g \) if \( f \circ g^{-1} \) is a convex function.

This concept of convexity, besides to classical convexity covers also log-convexity, i.e., logarithmically convexity (function \( f \) is log-convex if \( \log f \) is a convex function). Note that a function \( f \) is log-convex, if and only if \( \log f \) is convex with respect to \( f^{-1} \). Functions from Definition 15 have wide application in the statistics and the economics (see [7], [32]). The relationship between log-convex functions, the usual convex functions and multiplicatively convex function is considered in [22].

The following useful result is obtained in [10], [21].

**Theorem 16:** Let \( f, g \) be continuous, strictly increasing function on \([c, d]\) and \( \frac{f(x)}{g(x)} \) be defined throughout \((c,d)\). Then \( f \) is convex with respect to \( g \) if and only if
\[
\frac{g''(x)}{g'(x)} \leq \frac{f''(x)}{f'(x)}\] whenever \( x \in (c,d) \).

In [21], a geometric interpretation of previous result is given. Namely, if \( f \) is convex with respect to \( g \) on \([c,d]\),
$g$ strictly increasing, $f(c) = g(c)$ and $f(d) = g(d)$, then $f(x) \leq g(x)$ for $x \in [c, d]$.

We also will use the notion of subhomogeneous function with respect to another function.

**Definition 17:** If $f, g$ are continuous functions and $g$ is strictly monotonic, then $f$ is subhomogeneous with respect to $g$ if $f \circ g^{-1}$ is subhomogeneous function.

In this terminology and under the same assumptions for functions $\varphi$ and $g$ as in Theorem 12, $g \circ \varphi \circ g^{-1}$ is subhomogeneous if $g$ is subhomogeneous and $\varphi$ is subhomogeneous with respect to $g$, i.e., we have the following Lemma.

**Lemma 18:** Let $g$ and $\varphi$ be continuous, and $g$ strictly increasing subhomogeneous (respectively convex) function. If $\varphi$ is subhomogeneous (respectively convex) with respect to $g$, then $g \circ \varphi \circ g^{-1}$ is subhomogeneous (respectively convex) function.

**Example 19:** Let $\varphi(x) = x^2$, $g(x) = x^2 + 2x$, $x \in [0, \infty]$ then we have

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{1}{x} \quad \text{and} \quad \frac{g''(x)}{g'(x)} = \frac{1}{x + 1}$$

Obviously $\frac{\varphi''(x)}{\varphi'(x)} \leq \frac{g''(x)}{g'(x)}$ for $x > 0$. Due to Theorem 16 and (1),

$$g(\varphi(g^{-1}(x))) = 2(\sqrt{x + 1} - 1)^2 + (\sqrt{x + 1} - 1)^4$$

is subhomogeneous. Therefore, we have

$$\left(\sup_{x \in X} (f(x) \circ \psi(x))\right)^2 \leq \sup_{x \in X} \left(\psi(f(x))^2 \circ \psi(x)\right),$$

where function $\psi$ defines sup-measure $m$, $m(X) = e = \sqrt{2} - 1$ and $x \circ y = \sqrt{(2x + x^2)(2y + y^2) + 1}$.

An analogous theorem to Theorem 12 holds related to general semiring $([a, b], \sup, \circ)$, where $\circ$ is generated by an increasing and continuous function $g : [a, b] \to [0, \infty]$ for $[a, b] \subseteq [−\infty, \infty]$.

**Theorem 20:** Let $([a, b], \sup, \circ)$ be a semiring and $\circ$ be represented by an increasing generator $g$, $m$ be a complete sup-measure and $m(X) = e$, where $e$ is the neutral element of $\circ$. Let $\varphi : [a, b] \to [a, b]$ be continuous, strict increasing function such that

$$\varphi(x \circ y) \leq \varphi(x) \circ y \quad (7)$$

for all $y \in [a, e]$ and $x \in [a, b]$. Then for any measurable functions $f : X \to [a, b]$ the inequality (6) holds.

**Proof:** Let $\psi$ be the density function related to $m$. Due to the property of function $\varphi$ and (7) we have:

$$\varphi(\mathbf{P}s(m, f)) = \varphi\left(\sup_{x \in X} (f(x) \circ \psi(x))\right) = \sup_{x \in X} \varphi(f(x) \circ \psi(x)) \leq \sup_{x \in X} \varphi(f(x)) \circ \psi(x) = \mathbf{P}s(m, \varphi(f)).$$

which is the inequality we have to prove.

**Example 21:** Let $[−\infty, \infty], \sup, \circ$ be a semiring and $\circ$ be represented by a generator $g(x) = e^x$, i.e., $x \circ y = x + y$. The neutral element of $\circ$ is 0. Let $m$ be a complete sup-measure such that $m(X) = 0$. Then condition (7) is fulfilled when $\varphi(x) \leq x$ and $\varphi$ is subadditive function, i.e., $−\varphi$ is superadditive for every $x \in [−\infty, \infty]$. Since a necessary and sufficient condition that a measurable concave function $\varphi$ be subadditive is that $\varphi(0) \geq 0$ (see [15]), Jensen’s inequality (6) related to this semiring valid for concave function $\varphi$ such that $\varphi(x) \leq x$ for every $x \in [−\infty, \infty]$ and $\varphi(0) = 0$.

**V. Conclusion**

As a consequence of the Jensen type inequality for the smallest universal integral we get corresponding inequality for the pseudo-integral related to the semiring $([a, b], \sup, \circ)$, where $\circ$ is generated by an increasing and continuous function. We have observed the sufficient conditions for pseudo-operation under which that inequality holds. The corresponding inequality for more general universal integral will be the major focus of the further research.

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