A note on the use of Choquet and Sugeno integrals in Minimal and Maximal Covering Location Problems

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Abstract—The aim of this paper is to show a potential applicability of some well-known fuzzy integrals, i.e., of the Choquet integral and the Sugeno integral, in Minimal and Maximal Covering location problems, i.e., in MinCLP and MCLP. Possible benefits of the use of Choquet and Sugeno integrals lie in the flexibility of a monotone set function which is the core of the observed integrals and which is being used for modelling Decision Maker’s behavior. Mathematical models of Minimal and Maximal Covering location problems are given. Approach based on fuzzy integrals versus the standard two types of operators is discussed. Ideas for the future work and applications are presented.

Keywords: minimal, maximal, covering, Choquet integral, Sugeno integral.

I. INTRODUCTION

Maximal Covering Location Problems (MCLP) and Minimal Covering Location Problems (MinCLP) are well-known important combinatorial optimization problems.

The MCLP became interesting to computer scientist in the early 1970s when the need to optimally place facilities (hospitals, bus stations, ...) in order to maximize location coverage has occurred. A location is covered with at least one facility which is located within a given maximal distance. The research on this topic has developed in various directions ([5]).

However, the MinCLP have the opposite aim to the MCLP. The aim in the MinCLP is to find a location layout which has the minimal coverage. One of the real time problems would be the placement of pollution so that the environment pollution is minimal. Drezner and Wesolowsky (see [11]) introduced this MinCLP approach in 1990s.

If a partial coverage of locations is allowed, then Fuzzy MCLP and Fuzzy MinCLP are obtained. This yields to a different type of goal function calculation leading to a different quality of the solution. In FMCLP and FMinCLP solutions have more partially covered locations as opposed to the classical case where more location are completely covered but much more locations are left completely uncovered. Also, the introduction of fuzzyness does not increase the computational complexity of the algorithm. Many different meta-heuristics are used to solve these problems. Some examples are VNS (Variable Neighborhood Search), particle swarm optimization and simulated annealing. Based on the previous observations, it can be concluded that the particle swarm optimization (PSO) gives good results in solving these types of problems, therefore it is the chosen meta-heuristic for the current research.

Usually, t-norms and t-conorms ([18]) are used as aggregation operators in the calculation of the location coverage degree. However, the use of fuzzy integrals, namely of the Choquet integral and the Sugeno integral, can also measure the different interactions between locations yielding to a better-quality solution.

These two well-known types of integrals are based on an extremely wide class of set-functions, namely on fuzzy measures. Since the base for construction in both cases is a fuzzy measure, i.e., a monotone set-function, those integrals are highly flexible for modelling behavior of a Decision Maker. On the other hand, possible rigidity of the classical case, i.e., of the additivity that is essential for the classical measure is nicely illustrated with the Ellsberg’s paradox (see [13]). Also, some very effective examples of practical application of Choquet and Sugeno integrals can be found in [13], [23].

This paper is organized in the following manner. In Section 2 some basic notions regarding the Choquet integral and the Sugeno integral are given. The focus is on the discrete case. Section 3 contains the MCLP and MinCLP brief description of PSO. In the fourth section the mathematical model of the observed problem is given. Finally, Section 5 concludes this paper.

II. PRELIMINARIES - FUZZY INTEGRALS

As mentioned earlier, through this paper, as a tool for obtaining the measure of interactions between locations, which is being incorporated with a Decision Maker’s personal opinions, fuzzy integrals are being used. The focus of this section is on discrete case, i.e., on the discrete Choquet integral and on the discrete Sugeno integral since this discreetness is highly tangible in applications.
It has to be emphasized that these two discrete fuzzy integrals are highly applicable aggregation operators (see [10], [12]). The Choquet integral generalizes the so-called additive operators, e.g., the OWA (the ordered weighted averaging operators, see [25]) and the weighted mean, while the Sugeno generalizes the so-called "min-max" operators. Both cases give a "middleish" representative value.

The first necessary notion is the notion of a fuzzy measure. For this purpose, let \( X \) be a set of criteria, that is, let it be a set of all input values. Further on, let \( \mathcal{P}(X) \) be the power set of \( X \).

**Definition 2.1**: A set function on \( \mathcal{P}(X), \mu : \mathcal{P}(X) \to [0, \infty] \), that satisfies the following
- for arbitrary \( A, B \in \mathcal{P}(X), \) if \( A \subseteq B \) then \( \mu(A) \leq \mu(B) \) (monotonicity),
- is a non-decreasing set function, i.e., a fuzzy measure.

Now, the triplet \( (X, \mathcal{P}(X), \mu) \) is a fuzzy measure space ([23], [1], [2]). In general, instead of \( \mathcal{P}(X) \) a \( \sigma \)-algebra of subsets of \( X \) can be used.

**Remark 2.2**: For a fuzzy measure in the widest sense no additional assumptions are requested. However, if necessary, it is said that \( \mu \) is continuous from below if
\[
A_n \nearrow A \implies \lim_{n \to \infty} \mu(A_n) = \mu(A)
\]
for all sequences \( \{A_n\}_{n \in \mathbb{N}} \) from the power set. Similarly, it is said that \( \mu \) is continuous from above if
\[
A_n \searrow A \quad \text{and} \quad \mu(A_1) < \infty \implies \mu(A_n) \nearrow \mu(A)
\]
for all sequences \( \{A_n\}_{n \in \mathbb{N}} \) from the power set.

Some more strict approaches to a fuzzy measure contain continuity from below and from above as a part of the definition (see [23]).

Also, \( \mu \) is normalized if
\[
\mu(X) = 1.
\]

As already mentioned, for the purpose of this research the focus is on a discrete case, i.e., on simple functions. Therefore, further on, the following form of functions that will be observed
\[
f : X \to \{\omega_1, \omega_2, \ldots, \omega_h\},
\]
where \( \omega_i \in [0, \infty] \) and the working assumption, with no influence on generality, is
\[
0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_h.
\]

Even more, based on the type of problems that will be investigated in the future, it is sufficient to observe simple functions with values in \([0, 1]\) and normalized fuzzy-measures.

**A. Discrete Choquet integral**

The definition of the Choquet integral for the discrete case follows ([4]).

**Definition 2.3**: The Choquet integral of an arbitrary simple function \( f : X \to \{\omega_1, \omega_2, \ldots, \omega_h\} \), based on a fuzzy measure \( \mu \), is:
\[
\left(\int_X f \, d\mu\right) = \sum_{i=1}^{n} (\omega_i - \omega_{i-1}) \cdot \mu(\Omega_i)
\]
where \( \Omega_i = \{x \mid f(x) \geq \omega_i\} \) and \( \omega_0 = 0 \)

**Remark 2.4**: In general, for the continuous case the Choquet integral is given by the following:
\[
\left(\int_X f \, d\mu\right) = \int_0^\infty \mu(f > \omega) \, d\omega,
\]
where on the right-hand side is the classical Lebesgue integral.

Also, it is possible to extend the previous concept for functions \( f : X \to [-\infty, \infty] \). There are two well-known extensions, the symmetrical Choquet integral, also known as the ˇSipoš integral, and the asymmetric Choquet integral for which even monotonicity of the core set function can be omitted.

More on the Choquet integral can be found in, e.g., [1], [2], [4], [9], [13], [19].

**B. Discrete Sugeno integral**

The Sugeno integral for the discrete case is given by the following ([20]).

**Definition 2.5**: The Sugeno integral of an arbitrary function \( f : X \to \{\omega_1, \omega_2, \ldots, \omega_h\} \), based on a fuzzy measure \( \mu \), is:
\[
\left(\int_X f \, d\mu\right) = \max_{\alpha \in [0, \infty]} \left(\min_{\omega_i} (\mu(\Omega_i))\right)
\]
where \( \Omega_i = \{x \mid f(x) \geq \omega_i\} \).

**Remark 2.6**: The Sugeno integral for the general case is given by the following.

If \( f : X \to [0, \infty) \) is a measurable function, then
\[
\left(\int_X f \, d\mu\right) = \sup_{\alpha \in [0, \infty]} \left(\min_{\omega_i} (\mu(\Omega_i))\right),
\]
where \( \Omega_{\alpha} = \{x \mid f(x) \geq \alpha\} \).

In this case the convention \( \inf_{\omega_i \in \emptyset} f(x) = \infty \) is accepted.

More on Sugeno integral can be found in, e.g., [1], [2], [13], [19], [20], [23].

**C. Some examples**

Universality and wideness of fuzzy integrals as aggregation operators follow from minimal restrictions imposed on the background set function. This can be illustrated by the following example (see [10]).

**Example 2.7**: For the fuzzy measure \( \mu : \mathcal{P}(X) \to [0, 1] \) given by
\[
\mu(X) = 1 \quad \text{and} \quad \mu(A) = 0 \quad \text{for} \quad A \neq X,
\]
both corresponding fuzzy integrals coincide with the classical minimum.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(\emptyset) = 0 \quad \text{and} \quad \mu(A) = 1 \quad \text{for} \quad A \neq \emptyset, \]
  both corresponding fuzzy integrals coincide with the classical maximum.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(A) = 0 \quad \text{for} \quad \text{card}(A) \leq n-k \quad \text{and} \quad \mu(A) = 1 \quad \text{otherwise}, \]
  both corresponding fuzzy integrals coincide with the classical k-order statistic.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(A) = \frac{\text{card}(A)}{\text{card}(X)}, \]
  the corresponding Choquet integral coincides with the classical arithmetic mean.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(A) = \sum_{\omega_i \in A} \mu(\{\omega_i\}) \quad \text{and} \quad \mu(\{\omega_i\}) = w_i, \]
  where $w_i$ are pregiven weights, the corresponding Choquet integral coincides with the weighted mean.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(A) = \frac{\text{card}(A)-1}{\text{card}(X)} \sum_{j=0}^{n-1} w_{n-j}, \]
  where $w_i$ are pregiven weights, the corresponding Choquet integral coincides with the OWA operator.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(A) = 1 - \max_{\omega_i \in A} \mu(\{\omega_i\}) \quad \text{and} \quad \mu(\{\omega_i\}) = w_i, \]
  where $w_i$ are pregiven weights, the corresponding Sugeno integral coincides with the weighted minimum.
• For the fuzzy measure $\mu : \mathcal{P}(X) \rightarrow [0,1]$ given by
  \[ \mu(A) = \max_{\omega_i \in A} \mu(\{\omega_i\}) \quad \text{and} \quad \mu(\{\omega_i\}) = w_i, \]
  where $w_i$ are pregiven weights, the corresponding Sugeno integral coincides with the weighted maximum.

III. PARTICLE SWARM OPTIMIZATION

Many heuristics have been used to solve combinatorial optimization problems genetic algorithms, VNS... Lately, immunological algorithms are used [7]. In this paper the well known heuristics of Particle Swarm Optimization (PSO) will be used.

PSO is a popular tool often used to solve several optimization problems and it is formulated by Kennedy and Eberhart in 1995 in [14], [16]. It has been used in the electromagnetic community (see [26]). The Particle Swarm Optimization (PSO) is a popular tool often used to solve several optimization problems and it is formulated by Kennedy and Eberhart in 1995 in [15], [17]. This algorithm is inspired by the social behavior of individuals (particles) inside various swarms in the ecosystem (e.g. flocks of seagulls, schools of fish).

The classical PSO, analyzes a swarm $S$ that consists of $n$ particles $S = 1, 2, \ldots, n$ in a $d$-dimensional continuous space. The location of the $i$-th particle is at the position $x_i = x_{i1}, x_{i2}, \ldots, x_{id}$ and its velocity is $v_i = v_{i1}, v_{i2}, \ldots, v_{id}$. Now, the layout $x_i$ is a possible problem solution, meanwhile the velocity $v_i$ is the change of position of particle $i$ in the following iteration. Thus, particle $i$ in the iteration $k$ has the position $x_i^k = x_i^{k-1} + v_i^k$ which is a possible solution vector. Locations are the vector components, with a value 1 at the position $i$ if a facility $i$ is located at that particular location, and 0 otherwise.

Particle $i$ exchange information with its neighbors, $N(i) \subseteq S$, and resulting in a dynamical group variations. The idea is, that in the $k$-th iteration, particles $i$ update their velocities with according to their top value so far $(b_i)$ and their top value $(g_i)$ among its neighbors $N(i)$, using a well-known expression given in [15], [17]:

$$v_i^k = c_1 \xi v_i^{k-1} + c_2 \xi (b_i - x_i^{k-1}) + c_3 \xi (g_i - x_i^{k-1})$$  \hspace{1cm} (1)

The parameters $c_i$ are the degrees of confidence of particle $i$ in which are fluently interchangeable. The term $\xi$ is an uniformly distributed random value from a unit interval generated in each iteration.

This a discrete version of the PSO algorithm, many other implementations are given (see [15], [16], [17]).

IV. MATHEMATICAL MODEL OF MCLP AND MINCLP WITH THE CHOQUET AND SUGENO INTEGRAL

The mathematical model for the MinCLP is given in this section. Actually, this is a so-called constraint satisfaction problem model with partial coverage degrees with the Sugeno integral and the Choquet integral included in it.

The basics of the model are (the domains are in [ ] brackets):

- $P$ - number of facilities [integer];
- $X = \{L_1, L_2, \ldots, L_R\}$ - set of all locations
- $Y = \{Y_1, Y_2, \ldots, Y_R\} \subseteq X$ - set of all facilities
- $\mu : \mathcal{P}(X) \rightarrow [0,1]$ - interaction measure of different facilities modeled by a monotone set function.

Of course, the set of all facilities is a subset of the set of all locations.

Decision variables:

- $Y_i \in X$ - facility indicator $i$,
- $\omega_{i,j}$ - degree of coverage of the $j$-th facility from $[0,1]$ for location $L_i$.

The following formulae constitute the model:

$$f_{L_i} : X \rightarrow \{\omega_{i,1}, \omega_{i,2}, \ldots, \omega_{i,m}\},$$ \hspace{1cm} (2)

$$g = \sum_{i} (C) \int f_{Li} d\mu$$ \hspace{1cm} (3)

OR

$$g = \sum_{i} (S) \int f_{Li} d\mu.$$ \hspace{1cm} (4)
Namely, function (2) gives the degree of coverage of each node. Formulae (3) and (4) are actually the functions that are maximized (or minimized) in the model. If the function is maximized then the problem is MCLP and in the case that the function is minimized MinCLP is obtained. Moreover in MinCLP a minimal distance $d_{\text{min}}$ between two established facilities is given. If two facilities are closer then $d_{\text{min}}$ the solution is not acceptable.

The main difference between this model and the previous models is the introduction of the Choquet integral and the Sugeno integral. Previously, only the intersection of the fuzzy sets that represent the distance and the travel time between two nodes are considered. In this case the interaction of different nodes is considered in order to obtain a solution of a better-quality. This is can be achieved by predefining a measure $\mu$ on the partitive set of the nodes $\mathcal{P}(X)$ which measures the interaction of each sets of nodes.

Although the introduction of $\mu$ at the first glance seems to increase the computational complexity this is avoided because in the implementations only a few subsets are connected to a single node, thus the computational complexity is not increased. Some possible implementations will be investigated in the future work.

V. CONCLUSION

This is a pioneering work in the use of Choquet and Sugeno integrals in MCLP and MinCLP. The mathematical model is presented and also outlines of the PSO algorithm as one of the best meta-heuristics are given. Namely, we can see that the model is not complex making it very implementable. The future work will be focused on the implemented models and tests that will, hopefully, lead to a better-quality solutions. Finally, the proposed model covers most of the MCLP and MinCLP problems that have been analyzed in the previous years, therefore it can be viewed as a general Fuzzy CLP model.

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