On Multiplication of Interactive Fuzzy Numbers

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Abstract—In this paper we consider the extended product of two positive interactive fuzzy numbers and will give sufficient and necessary conditions for the equality of their interactive and non-interactive products.

I. CONCEPTS AND ISSUES

A fuzzy number $A$ (see e.g. [13]) is a fuzzy subset in $\mathbb{R}$ with a convex upper semicontinuous membership function of bounded support. The family of fuzzy numbers is denoted by $\mathcal{F}$. Let us recall that if $\mathcal{X}$ is a metric space then a function $f$ is called upper semicontinuous on $\mathcal{X}$ if

$$
\limsup_{x \to x_0} f(x) \leq f(x_0)
$$

for any $x_0 \in \mathcal{X}$. The convexity of $A$ is equivalent with $A(\alpha x + (1-\alpha)y) \geq \min\{A(x), A(y)\}$ for all $\alpha \in [0,1]$ and $x, y \in \mathbb{R}$. This is equivalent to say that $A$ is a quasi-concave function.

In what follows we recall well known results about the $\gamma$-cut representation of fuzzy numbers (see e.g. [8]).

If $A$ denotes a fuzzy number and $\gamma \in (0,1]$, the crisp set given by

$$[A]^\gamma = \{x \in \mathbb{R} : A(x) \geq \gamma\},$$

is called the $\gamma$-cut of $A$. Then, the set

$$[A]^0 = \{x \in \mathbb{R} : A(x) > 0\},$$

is called the support of $A$. We denote $A_0 = \text{supp}(A)$.

The set $[A]^\gamma$ is called the core of $A$ which will be denoted in this paper with core$(A)$. If core$(A)$ is reduced to a single point then often $A$ is called a unimodal fuzzy number and in this case the value core$(A)$ is called modal value. Often fuzzy numbers which are not unimodal are called fuzzy intervals (see e.g. [8]).

It is easily seen that every $\gamma$-cut $\gamma \in (0,1]$, of the fuzzy number $A$ is a closed interval

$$[A]^\gamma = [a_1(\gamma), a_2(\gamma)],$$

where

$$a_1(\gamma) = \inf\{x \in \mathbb{R} : A(x) \geq \gamma\},$$

$$a_2(\gamma) = \sup\{x \in \mathbb{R} : A(x) \geq \gamma\}$$

for any $\gamma \in (0,1]$. Denoting $[A]^0 = [a_1(0), a_2(0)]$, we obtain a parametric representation $(a_1, a_2)$ of the fuzzy number $A$ and it is well known that the function $a_1 : [0,1] \to \mathbb{R}$ is non-decreasing and left continuous, while the function $a_2 : [0,1] \to \mathbb{R}$ is left continuous and non-increasing (see Theorem 1.1 in [7] where actually the authors prove the left continuity on $(0,1]$ and the continuity at $0$ but anyway this is equivalent with the left continuity on $[0,1]$).

Suppose that $\text{supp}(A) = [a, b]$ and core$(B) = [c, d]$. It results that $A(x) = 0$ for all $x \in (-\infty, a)$ and $0 < A(x) < A(c)$ for all $x \in (a, c)$. If $A$ is continuous at $a$ then we have $A(a) = 0$ otherwise we have $A(a) > 0$. The restriction of $A$ to the interval $[a, c]$ is often called in the literature as the left side (branch) of the fuzzy number $A$ and in this paper we will use for it the notation $l_A$. Similarly, the right side of $A$ denoted by $r_A$ represents the restriction of $A$ to $[d, b]$.

Finally, we notice that in general, by the upper semicontinuity of $A$ and by the definitions of $a_1$ and $a_2$ respectively, it is easily seen that

$$A(a_1(\gamma)) \geq \gamma, \forall \gamma \in [0,1]$$

$$A(a_2(\gamma)) \geq \gamma, \forall \gamma \in [0,1].$$

A two-dimensional joint possibility distribution $C$ (see [5]) is a fuzzy set in $\mathbb{R}^2$ such that its projections,

$$A(x) = \max_{z \in \mathbb{R}} C(x, z), \ B(y) = \max_{z \in \mathbb{R}} C(z, y), \ x, y \in \mathbb{R},$$

are fuzzy numbers.

The extension principle is one of the most important concepts in the theory of fuzzy sets, where we extend crisp (non-fuzzy) domains of mathematical functions to fuzzy domains. Using the concept of joint possibility distribution Carlsson, Fullér and Majlender [2] introduced the following sup-$C$ extension principle in 2004.

Definition I.1 (Carlsson, Fullér and Majlender, [2], Definition 2.1). Let $C$ be a joint possibility distribution with marginal possibility distributions $A_1, A_2, \ldots, A_n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then $f_C(A_1, \ldots, A_n) \in \mathcal{F}$, given by

$$f_C(A_1, \ldots, A_n)(y) = \sup_{f(x_1, \ldots, x_n) = y} C(x_1, \ldots, x_n)$$

is the interactive extension of $f$ with respect to $C$. In addition $f_C(A_1, \ldots, A_n) \in \mathcal{F}$.

Just above we mentioned that $f_C(A_1, \ldots, A_n)$ is a fuzzy number which is based on the fact that this property was proved in Lemma 2.1 of the paper [2]. Fuzzy numbers $A, B \in \mathcal{F}$ are said to be non-interactive if their joint possibility distribution is defined by $A \times B$ [5].
We should note here that if
\[ C(x, y) = \min\{A(x), B(y)\} \]
where \( A, B \in \mathcal{F} \) then (3) turns into the sup-min extension principle introduced by Zadeh [12]. Furthermore, if
\[ C(x, y) = T(A(x), B(y)), \]
where \( T \) is a t-norm [11] then we get the t-norm-based extension principle.

**II. The Extension Principle for Interactive Fuzzy Numbers**

Based on Definition I.1, the interactive product of fuzzy numbers \( A \) and \( B \) with respect to their joint possibility distribution \( C \) denoted with \( A \odot_C B \), is given by
\[ (A \odot_C B)(z) = \sup_{x, y = z} C(x, y), \forall z \in \mathbb{R}. \]

If \( C \) is upper semicontinuous on \( \text{supp}(A) \times \text{supp}(B) \) then we actually have
\[ (A \odot_C B)(z) = \max_{x, y = z} C(x, y), \forall z \in \mathbb{R}. \]  \( \text{(4)} \)

Indeed, supposing that \( \text{supp}(A) = [a, b] \) and \( \text{supp}(B) = [a', b'] \), first of all we notice that by the definition of \( A \odot_C B \) we have \( A \odot_C B \leq A \cdot B \) because for any \( x, y, z \in \mathbb{R} \) such that \( x \cdot y = z \), we have
\[ C(x, y) \leq A(x) \wedge B(y) \leq (A \cdot B)(z) \]
and taking the supremum over all \( (x, y) \) we get
\[ (A \odot_C B)(z) \leq (A \cdot B)(z). \]

This easily implies that \( (A \odot_C B)(z) = 0 \) for any \( z \) which is outside \( \text{supp}(A \cdot B) \) and therefore in these cases there is nothing to be proved. If \( z \in \text{supp}(A \cdot B) \) then obviously we can write
\[ (A \odot_C B)(z) = \sup_{(x, y) \in K_z} C(x, y), \]
where
\[ K_z = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], y \in [a', b'], x \cdot y = z\}. \]

Since obviously \( K_z \) is a compact subset of \( \mathbb{R}^2 \), combining the upper semicontinuity of \( C \) with the extreme value theorem for upper semicontinuous functions, we obtain that (4) holds.

In what follows, we will compute the interactive product of two fuzzy numbers using concrete joint possibility distributions. We mention that the first two joint possibility distributions are well known in the literature (numerous examples including those presented below can be found in Chapter 3 of the book [1]) although, so far, they were not used to compute interactive multiplications.

Let us consider fuzzy numbers \( A, B \) where
\[ A(x) = B(x) = (1 - x)\chi_{[0,1]}(x), \]
It is easily seen that \( [A]^\gamma = [B]^\gamma = [0, 1 - \gamma], \gamma \in [0, 1] \). First let us consider the joint possibility distribution generated by the weakest triangular norm
\[ W: [0, 1]^2 \rightarrow [0, 1] \]
where \( W(x, y) = \min\{x, y\} \) when \( \max\{x, y\} = 1 \) and \( W(x, y) = 0 \) otherwise. Then \( C_W: \mathbb{R}^2 \rightarrow \mathbb{R} \),
\[ C_W(x, y) = W(A(x), B(y)) \]
is a joint possibility distribution of \( A \) and \( B \) which is generated by \( W \). It is immediate that we have
\[ C_W(x, y) = \begin{cases} A(x) & \text{if } y = 0 \\ B(y) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}. \]

Applying formula (4) for some \( z \in \mathbb{R} \), we obtain
\[ (A \odot_{C_W} B)(z) = \max_{x, y = z} W(A(x), B(y)). \]

Obviously, if \( z \neq 0 \) then by the definition on \( C_W \) we easily obtain \( (A \odot_{C_W} B)(z) = 0 \). Otherwise, if \( z = 0 \) then it is immediate that \( (A \odot_{C_W} B)(z) = 1 \). Hence, \( (A \odot_{C_W} B)(z) = 1 \). It worth to be noticed that multiplication based on the weakest triangular norm is the only triangular norm based multiplication which preserves the linear shape of the fuzzy numbers (see [9]).

Now let us consider the joint distribution of \( A \) and \( B \) given by \( C: \mathbb{R}^2 \rightarrow \mathbb{R}, C(x, y) = (1 - x - y)\chi_P(x, y), \) where \( P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\} \). Firstly, if \( z \notin [0, 1] \) we notice that for any pair \( (x, y) \in \mathbb{R}^2 \) which satisfies \( x \cdot y = z \), we have \( (x, y) \notin P \) and therefore this implies \( (A \odot_C B)(z) = 0 \). Before we discuss the case \( z \in [0, 1] \), it will be more convenient to use the \( \gamma \)-cut representation of \( C \) which is (see e.g. [6], pp. 55)
\[ [C]^\gamma = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1 - \gamma\}, \gamma \in [0, 1]. \]

Considering the function \( f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = x \cdot y, \) and applying for it the conclusion of Lemma 2.1 from paper [2], it results that \( [A \odot_C B]^\gamma = f([C]^\gamma), \gamma \in [0, 1] \). So, let us fix arbitrary \( \gamma \in [0, 1] \). It is immediate that \( \min(f([C]^\gamma)) = 0 \). Then if \( (x, y) \in [C]^\gamma \) we observe that
\[ f(x, y) = x \cdot y \leq \frac{(1 - \gamma)}{2} \cdot \frac{(1 - \gamma)}{2} \]
and since
\[ \left(\frac{(1 - \gamma)}{2}, \frac{(1 - \gamma)}{2}\right) \in [C]^\gamma, \]
we get that
\[ \max(f([C]^\gamma)) = \frac{(1 - \gamma)^2}{4}. \]
Thus,
\[ [A \odot_C B]^\gamma = \left[0, \frac{(1 - \gamma)^2}{4}\right], \gamma \in [0, 1] \]
and by elementary calculus this implies that
\[ (A \odot_C B)(z) = 0 \text{ if } z \in (-\infty, 0) \cup [1/4, \infty) \]
and
\[ (A \odot_C B)(z) = 1 - 2\sqrt{z} \text{ if } z \in [0, 1/4). \]
Next, we consider the joint distribution of $A$ and $B$ given by $C : \mathbb{R}^2 \to \mathbb{R}$,
$$C(x, y) = (1 - x)\chi_S(x, y),$$
where
$$S = \{(x, x) \in \mathbb{R}^2 : x \in [0, 1]\}.$$  
It is immediate that $$(A \circ C)B(z) = 0$$ for all $z$ outside $[0, 1]$. Thus, we can prove that if $z \in [0, 1]$, then
$$(A \circ C)B(z) = \max_{x \leq y \leq z} C(x, y) = \max_{x \leq y \leq z} (1 - x)\chi_S(x, y),$$
which easily implies $$(A \circ C)B(z) = 1 - \sqrt{z}.$$ Interestingly, $A \circ C B = A \cdot B$ where $A \cdot B$ denotes standard multiplication according to the Zadeh’s extension principle as this can be easily checked.

Let us analyze deeper the last example. We observe that the definition of $C$ implies that $C(a_2(\gamma), b_2(\gamma)) = \gamma$ for any $\gamma \in [0, 1]$. In what follows we will prove that such property is sufficient in order to have the equality between the standard and interactive product of positive fuzzy numbers. Let us consider two arbitrary positive fuzzy numbers $A$ and $B$. The diagonal of the pair $(A, B)$ is the set (see [4], Definition 5) $\Psi(A, B) = \Psi_1 \cup \Psi_2 \cup \Psi_3$, where
$$\Psi_1 = \{ x = a_1(\gamma), y = b_1(\gamma), \gamma \in [0, 1]\},$$
$$\Psi_2 = \{ (a_1(1), b_1(1)), (a_2(1), b_2(1)) \},$$
$$\Psi_3 = \{ x = a_2(\gamma), y = b_2(\gamma), \gamma \in [0, 1]\}.$$ 
Here, $\{ (a_1(1), b_1(1)), (a_2(1), b_2(1)) \}$ is the line which units the points $(a_1(1), b_1(1))$ and $(a_2(1), b_2(1))$ in $\mathbb{R}^2$. We notice that if $A$ and $B$ are unimodal fuzzy numbers (that is the core is reduced to a single element) then $\Psi(A, B) = \Psi_1 \cup \Psi_2$. We also need the sets $\Psi'_1 \subseteq \Psi_1$ and $\Psi'_3 \subseteq \Psi_3$, where
$$\Psi'_1 = \{ x = a_1(\gamma), y = b_1(\gamma), \gamma \in \text{Im}(l_{A\cdot B})\},$$
$$\Psi'_3 = \{ x = a_2(\gamma), y = b_2(\gamma), \gamma \in \text{Im}(r_{A\cdot B})\}.$$ 
Here, if $f : A \to B$, then $\text{Im}(f) = f(A) = \{ f(x) : x \in A \}$. As a result of independence importance it worth to be mentioned that it can be proved that if $A$ and $B$ are positive fuzzy numbers, then actually we have $\text{Im}(l_{A \cdot B}) = \text{Im}(I_A) \cup \text{Im}(l_B)$ and $\text{Im}(r_{A \cdot B}) = \text{Im}(I_A) \cup \text{Im}(r_B)$. Indeed, we just need to repeat the reasoning from the proof of Proposition 7 in [4] where the conclusion is the same when we consider addition of fuzzy numbers, that is $\text{Im}(l_{A + B}) = \text{Im}(l_A) \cup \text{Im}(l_B)$ and $\text{Im}(r_{A + B}) = \text{Im}(r_A) \cup \text{Im}(r_B)$. Practically the proof of this proposition is based on the fact that
$$(A + B)^\gamma = [a_1(\gamma) + b_1(\gamma), a_2(\gamma) + b_2(\gamma)], \gamma \in [0, 1].$$
Or, it is easy to check that we can perform the same reasonings for positive fuzzy numbers $A$ and $B$ noting that now we have
$$(A \cdot B)^\gamma = [a_1(\gamma) \cdot b_1(\gamma), a_2(\gamma) \cdot b_2(\gamma)], \gamma \in [0, 1].$$
Thus, $\Psi'_1$ and $\Psi'_3$ are the same as in [4] since
$$\text{Im}(l_{A \cdot B}) = \text{Im}(l_{A+B}) = \text{Im}(l_A) \cup \text{Im}(l_B)$$
and
$$\text{Im}(r_{A \cdot B}) = \text{Im}(r_{A+B}) = \text{Im}(r_A) \cup \text{Im}(r_B).$$
Now, suppose that $C : \mathbb{R}^2 \to \mathbb{R}$ is a joint distribution with marginal distributions being the positive fuzzy numbers $A$ and $B$ and, suppose that for any $\gamma \in \text{Im}(l_{A \cdot B})$ we have
$$C(a_1(\gamma), b_1(\gamma)) = \gamma$$
and for any $\gamma \in \text{Im}(r_{A \cdot B})$ we have
$$C(a_2(\gamma), b_2(\gamma)) = \gamma.$$ Then it holds that $A \circ C B = A \cdot B$ as it will be proved in what follows.

Since
$$(A \circ C B) \leq A \cdot B,$$
it suffices to prove that
$$(A \circ C B)(z') = \sup_{x \cdot y = z'} C(x, y) \geq C(a_1(\gamma), b_1(\gamma)) = \gamma.$$ On the other hand, since $z' \leq z$, noting that $A \cdot B$ is nondecreasing on $[z', z]$ (this is so because $z' \leq z \leq \min(\text{core}(A \cdot B))$), it necessarily results that $A \circ C B$ is nondecreasing on $[z', z]$ (otherwise there would exist $z' \leq t < z$ such that $(A \circ C B)(t) = 1$ and since $t < \min(\text{core}(A \cdot B))$ we would obtain the contradiction $(A \cdot B)(t) < (A \circ C B)(t)$). Thus, we have $\gamma = (A \circ C B)(z') \leq (A \circ C B)(z)$. But since $(A \circ C B)(z) \leq (A \cdot B)(z)$, it results that in fact we must have $(A \circ C B)(z) = (A \cdot B)(z) = \gamma$. By similar reasonings we get the same equality in the case when $z \geq \max(\text{core}(A \cdot B))$. Therefore, we have just proved that
$$(A \circ C B)(z) = (A \cdot B)(z)$$
for all $z \in \text{supp}(A \cdot B) \setminus \text{int}(\text{core}(A \cdot B))$ which together with the fact that $A \circ C B \leq A \cdot B$, results in the equality $A \circ C B = A \cdot B$. Summarizing, a sufficient condition for the equality $A \circ C B = A \cdot B$ is to have $C(a_1(\gamma), b_1(\gamma)) = \gamma$ for any $\gamma \in \text{Im}(l_{A \cdot B})$ and $C(a_2(\gamma), b_2(\gamma)) = \gamma$ for any $\gamma \in \text{Im}(r_{A \cdot B})$. Let us now give an interpretation of this result by using the sets $\Psi'_1$ and $\Psi'_3$ introduced just above. First of all let us notice that the property $C(a_1(\gamma), b_1(\gamma)) = \gamma$ for all $\gamma \in \text{Im}(l_{A \cdot B})$ is equivalent with
$$C(x, y) = C_{id}(x, y) = A(x) \wedge B(y).$$
for all \((x, y) \in \Psi_1\). For this, it suffices to prove that \(C(x, y) = A(x) \wedge B(y)\) for any \((x, y) \in \Psi_1\). Therefore let us choose arbitrary \((x, y) \in \Psi_1\) and let \(r \in \mathbb{R}\) be such that \(x = a_1(\gamma)\) and \(y = b_1(\gamma)\). At first we notice that \(A(a_1(\gamma)) \geq \gamma\) and \(B(b_1(\gamma)) \geq \gamma\) (see (1)). But since \(r \in \text{Im}(l_{A,B})\) it results that \(\gamma \in \text{Im}(l_{A,B})\). Without any loss of generality we may assume that \(\gamma \in \text{Im}(l_A)\). It is a simple exercise to prove that in these conditions we actually have \(A(a_1(\gamma)) = \gamma\) (see also Proposition 6 in [4]). Thus, we get that
\[
A(x) \wedge B(y) = (A(a_1(\gamma))) \wedge B(b_1(\gamma)) = \gamma = C(x, y).
\]
for all \((x, y) \in \Psi_1\). So, if \(C\) and \(C'\) both take the same values through the set \(\Psi_1' \cup \Psi_3'\) then \(A \circ C = A \cdot B\). In particular, if \(C\) and \(C'\) both take the same values through the set \(\Psi(A, B)\) then the conclusion is the same.

Under some additional assumptions, the above conditions on \(C\) are also necessary. We assume that \(A\) and \(B\) are strictly positive which obviously implies that \(A \cdot B\) is strictly positive too. Then suppose that \(C\) is upper semicontinuous on \(\text{supp}(A) \times \text{supp}(B)\) and suppose the functions \(l_{A,B}\) and \(r_{A,B}\) are strictly monotone. It can be proved (reasoning as in the proof of Proposition 8 in [4]) that this is equivalent to say that \(l_{A,B}, r_{A,B}, r_A\) and \(r_B\) are strictly monotone. Then suppose that \(A \circ C = A \cdot B\). Now, let us choose arbitrary \(\gamma \in \text{Im}(l_{A,B})\).

We will prove that \(C(a_1(\gamma), b_1(\gamma)) = \gamma\). Since \(\gamma \in \text{Im}(l_{A,B})\), there exists \(z \leq \min(\text{core}(A \cdot B))\) such that \(l_{A,B}(z) = \gamma\). The fact that \(l_{A,B}\) is strictly monotone implies that we have \(z = \min((A \cdot B)^-\gamma)\) and since both \(A\) and \(B\) are positive fuzzy numbers we get that \(z = a_1(\gamma) \cdot b_1(\gamma)\). Then let \((x, y) \in \mathbb{R}^2\) be such that \(x \cdot y = z\) and \((A \circ C)(z) = C(x, y)\) (obviously such pair exists since by the upper semicontinuity of \(C\) we can use formula (4)). From the inequalities
\[
A(x) \wedge B(y) \leq (A \cdot B)(z)
\]
we get that
\[
(A \circ C)(z) = (A \cdot B)(z) = C(x, y) = A(x) \wedge B(y).
\]

Firstly, we discuss the case when \(\gamma > 0\). Therefore, we necessarily have \(x \in \text{supp}(A)\) and \(y \in \text{supp}(B)\) which implies that \(x\) and \(y\) are both positive. Since \(x \cdot y = a_1(\gamma) \cdot b_1(\gamma)\) it results that \(x \leq a_1(\gamma)\) or \(y \leq b_1(\gamma)\). Without any loss of generality suppose that \(x \leq a_1(\gamma)\). By way of contradiction suppose that in fact we have \(x < a_1(\gamma)\). Noting that \(\gamma > 0\), we obtain \(A(x) < \gamma\) which easily results in \(A(x) \wedge B(y) < A(a_1(\gamma)) \wedge B(b_1(\gamma))\). Since clearly \(A(a_1(\gamma)) \wedge B(b_1(\gamma)) \leq (A \cdot B)(z)\) we obtain \(A(x) \wedge B(y) < (A \cdot B)(z)\), a contradiction. Therefore, we must have \(x = a_1(\gamma)\). Since \(z > 0\) (recall \(A \cdot B\) is strictly positive because otherwise, if \(A \cdot B\) is not strictly positive then we can have \(x = 0\) and therefore it not necessarily results that \(y = b_1(\gamma)\) and \(x \cdot y = a_1(\gamma) \cdot b_1(\gamma)\), it results that we also have \(y = b_1(\gamma)\). Thus, we obtain the desired conclusion in this case, that is \(C(a_1(\gamma), b_1(\gamma)) = \gamma\). It remains to discuss the case when \(\gamma = 0\). Then necessarily we have
\[
z = a_1(0) \cdot b_1(0)
\]
because there is no other possibility to obtain \(l_{A,B}(z) = 0\). So, since \((A \cdot B)(a_1(0) \cdot b_1(0)) = 0\) it results \(A(a_1(0)) \wedge B(b_1(0)) = 0\). Without any loss of generality suppose that \(A(a_1(0)) = 0\). Thus, we obtain
\[
C(a_1(0), b_1(0)) \leq A(a_1(0)) \wedge B(b_1(0)) = 0.
\]
and therefore we obtain the desired conclusion, that is \(C(a_1(0), b_1(0)) = 0\). Summarizing, we have just proved that \(C(a_1(\gamma), b_1(\gamma)) = \gamma\) for all \(\gamma \in \text{Im}(l_{A,B})\). By absolute similar reasonings we obtain that \(C(a_2(\gamma), b_2(\gamma)) = \gamma\) for all \(\gamma \in \text{Im}(r_{A,B})\).

In other words we have just proved that
\[
C(x, y) = C_{id}(x, y) = A(x) \wedge B(y)
\]
for all \((x, y) \in \Psi_1' \cup \Psi_3'\).

In the case when \(A\) or \(B\) is just positive without being strictly positive then from the above reasonings it results that \(C(a_1(\gamma), b_1(\gamma)) = \gamma\) for all \(\gamma \in \text{Im}(l_{A,B})\setminus\{l_{A,B}(0)\}\) and \(C(a_2(\gamma), b_2(\gamma)) = \gamma\) for all \(\gamma \in \text{Im}(r_{A,B})\). Here we assume that \(A\) and \(B\) are not crisp numbers (these kinds of fuzzy numbers do not present interest for interactive fuzzy arithmetic) which implies that the domain of the function \(r_{A,B}\) does not contain the value 0.

All what we have discussed above can be summarized in the following theorem which can be considered the analogous of Theorem 12 in [4] for the case of interactive addition.

**Theorem II.1.** Let \(A\) and \(B\) denote two positive fuzzy numbers and consider \(C\) a joint possibility distribution of \(A\) and \(B\). If \(C(a_1(\gamma), b_1(\gamma)) = \gamma\) for all \(\gamma \in \text{Im}(l_{A,B})\) and \(C(a_2(\gamma), b_2(\gamma)) = \gamma\) for all \(\gamma \in \text{Im}(r_{A,B})\), then \(A \circ C = A \cdot B\). If in addition \(A\) and \(B\) are strictly positive, \(l_{A,B}\) and \(r_{A,B}\) are strictly monotone and \(C\) is upper semicontinuous on \(\text{supp}(A) \times \text{supp}(B)\), then the converse implication holds too.

By using the independent joint possibility distribution we have the following equivalent formulation.

**Corollary II.1.** Let \(A\) and \(B\) denote two positive fuzzy numbers with continuous membership functions and with strictly monotone sides. If \(C\) is a joint possibility distribution of \(A\) and \(B\) such that
\[
C(a_1(\gamma), b_1(\gamma)) = C(a_2(\gamma), b_2(\gamma)) = \gamma
\]

for all $\gamma \in [0, 1]$, then $A \circ_C B = A \cdot B$. If in addition $A$ and $B$ are strictly positive and $C$ is upper semicontinuous on $\text{supp}(A) \times \text{supp}(B)$ then the converse implication holds too.

By using the independent joint possibility distribution we have the following equivalent formulation.

**Corollary II.3.** Let $A$ and $B$ denote two positive fuzzy numbers with continuous membership functions and with strictly monotone sides. If $C$ is a joint possibility distribution of $A$ and $B$ such that

$$C(x, y) = C_{id}(x, y)$$

for any $(x, y) \in \Psi_1 \cup \Psi_3$ then

$$A \circ_C B = A \cdot B.$$ 

If in addition $A$ and $B$ are strictly positive and $C$ is upper semicontinuous on $\text{supp}(A) \times \text{supp}(B)$, then the converse implication holds too.

### III. Summary

The key element in the study of the equality

$$A \circ_C B = A \cdot B$$

is to analyse the values of $C$ throughout the diagonal of $A$ and $B$. Similarly to the case of interactive addition (studied in detail in [4]), if the values of $C$ coincide with the values of $C_{id}$ on the diagonal of $A$ and $B$ then we have $A \circ_C B = A \cdot B$ assuming that $A$ and $B$ are positive fuzzy numbers. We guess that the same equality holds for any interactive extension as long it is generated by a function $f : \mathbb{R}^{n} \to \mathbb{R}$ which is strictly increasing in each argument restricted to the support of the corresponding fuzzy number. Such result requires a more rigorous approach (at first impression it seems that we will need to use Proposition 5.1 from paper [10] and some quite technical auxiliary results) and this will be done in the future.

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